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Reduction of degenerate Lagrangian systems

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Abstract. *The geometrical structure of (finite dimensional) degenerate Lagrangian systems is studied and a reduction scheme, leading to a regular Lagrangian description of these systems on a reduced velocity phase space, is developed. The connection with the canonical approach to the regularization problem of degenerate systems (Dirac's theory) and the reduction of systems with symmetry (Marsden-Weinstein theory) is investigated. Some examples and applications are discussed.*

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1. INTRODUCTION

The aim of this paper is to analyse the geometrical structure of degenerate Lagrangians and to investigate the possibility of establishing a consistent regular Lagrangian description for the associated dynamical systems. More precisely, we will address the following question: given a degenerate Lagrangian L defined on some velocity phase space (i.e. a tangent bundle) is it possible to construct another velocity phase space and a regular Lagrangian which contains the same dynamical information as L? We may refer to this problem as the «regularization problem» for degenerate Lagrangians.

Conditions will be found which guarantee the existence of such a regularization and a relevant class of Lagrangians will be identified for which the above question admits an affirmative answer.

The starting point of our analysis will be the natural reduction of the original velocity phase space with respect to the characteristic distribution of the presymplectic 2-form ω_L associated with the given degenerate Lagrangian L. The reduced space is then the obvious candidate for the new velocity phase space, provided it can be equipped with an integrable almost tangent structure. Our approach to the regularization problem turns out to be interesting for physical as well as mathematical reasons.

From the physical point of view, degenerate Lagrangians are extremely important and are used for the description of basic physical theories (gauge theories). There exists a canonical approach to the regularization problem which traces back to the work of P.A.M. Dirac [1] and which has recently been completed with the development of the presymplectic constraint algorithm (cf. M. Gotay et al. [2]) and the Marsden-Weinstein reduction theory [3], where the zero level set of some momentum map serves to identify the constraints of the problem. In spite of this, the modern approach to the quantization of field theories is based on the Lagrangian rather than the Hamiltonian formulation of these theories. It therefore seems useful to develop a complete Lagrangian theory for the reduction of degenerate systems. Previous work in this direction can be found for instance in [4], [5] and [6]. One of the merits of a pure Lagrangian treatment of degenerate systems is also to be found in its contribution to a complete understanding of the real nature of gauge transformations, which play a central role in physical theories and which, in the canonical picture, have been identified with transformations generated by first class constraints.

From the mathematical point of view, the geometry of a Lagrangian on a tangent bundle has many interesting features. First of all, the presymplectic structure ω_L associated with L and the natural integrable almost tangent structure of the tangent bundle are, in a certain sense, «intertwining» operators. In particular, this entails a remarkable relation between the dimension of the full characteristic distribution of ω_L and the dimension of its vertical part, from which one may deduce a natural classification of Lagrangians. In developing the Lagrangian reduction theory, we first establish conditions under which the reduced symplectic manifold, obtained by taking the quotient of the original tangent bundle with respect to the characteristic distribution of ω_L, inherits an integrable almost tangent structure. We then further investigate when the given degenerate Lagrangian induces a regular Lagrangian system on the reduced «velocity phase space».

Strictly speaking, this reduced space will not be (diffeomorphic to) a genuine tangent bundle in general. It will be pointed out, however, that the concept of a Lagrangian system can be extended in a natural way to the broader framework of manifolds which posses an integrable almost tangent structure.

A specific class of Lagrangians is singled out for which the proposed reduction scheme works very nicely, namely those Lagrangians for which the associated characteristic distribution is spanned by the complete and vertical lift of a distribution defined on the base manifold (configuration space of the system).

The relation between the Lagrangian reduction theory and the usual canonical reduction procedures is investigated. A key-role is thereby assigned to the Legendre map. This analysis in particular reveals a close connection between the reduction of degenerate Lagrangian systems and the Marsden-Weinstein reduction of Hamiltonian systems with symmetry. A clear illustration of this link is obtained for systems of mechanical type (i.e. systems with a Hamiltonian of the form kinetic plus potential energy).

The paper is organized as follows. In section 2 we develop the Lagrangian reduction theory after giving a brief summary of some relevant facts from the geometry of Lagrangian dynamics. Section 3 is devoted to the relation between the Lagrangian and the canonical reduction procedures. Some examples and applications are discussed in section 4. We conclude with a few general comments in section 5. Two appendices are devoted to the proofs of some intermediate results which are also of some interest in their own. In the present paper we confine ourselves to finite dimensional systems. The infinite dimensional case, important for physical applications, will be considered in later work. All objects (mappings, vector fields, forms . . .) are assumed to be of class C^∞.

2. REDUCTION OF DEGENERATE LAGRANGIAN SYSTEMS

2.1. Geometry of Lagrangian dynamics

We start with a brief review of some basic facts from the tangent bundle geometry for Lagrangian dynamics (see e.g. [7], [8]).

Let Q be a finite dimensional differentiable manifold with tangent bundle

TQ, also called velocity phase space, and tangent bundle projection $\tau_Q: TQ \to Q$. Natural bundle coordinates on TQ are denoted by (q^i, v^i).

The bundle of vertical tangent vectors $V(TQ)$ is the subbundle of $T(TQ)$ defined by

$$V(TQ) = \{\xi \in T(TQ) \mid \tau_{Q*}(\xi) = 0\}.$$

On TQ there exists a canonical type (1,1) tensor field S, sometimes called the vertical endomorphism, which defines the natural almost tangent structure on TQ and which is characterized by

(i) $\operatorname{Im}(S) = \operatorname{Ker}(S) = V(TQ)$,

(ii) the Nijenhuis tensor N_S of S vanishes.

(cf. [7], [8], [9]).

Recall that the Nijenhuis tensor of a type (1,1) tensor field R on a manifold is a type (1,2) tensor field N_R which is defined by its action on vector fields as follows

$$N_R(X, Y) = R^2([X, Y]) + [R(X), R(Y)] - R([R(X), Y]) - R([X, R(Y)]).$$

The first of the above properties of S immediately implies $S^2 = 0$. In terms of natural bundle coordinates on TQ, S reads

$$S = \frac{\partial}{\partial v^i} \otimes \mathrm{d}q^i.$$

The dilation vector field on TQ, which is the generator of the 1-parameter group of dilations $(q, v) \to (q, e^t v)$, is denoted by Δ. In coordinates,

$$\Delta = v^i \frac{\partial}{\partial v^i}.$$

A vector field Γ on TQ is called a second-order equation field if $S(\Gamma) = \Delta$. This means that Γ is locally of the form $\Gamma = v^i\, \partial/\partial q^i + \Lambda^i(q, v)\, \partial/\partial v^i$. To each smooth function L on TQ one can assign a 1-form θ_L defined by

$$\theta_L = \mathrm{d}L \circ S.$$

L is called a Lagrangian and θ_L its Poincaré-Cartan 1-form. If we put

$$\omega_L = \mathrm{d}\theta_L,$$

then the following fundamental relation exists between the 2-form ω_L and the canonical tensor field S:

$$S(X) \lrcorner\, \omega_L = -(X \lrcorner\, \omega_L) \circ S, \tag{2.1}$$

for each vector field X on TQ (cf. [8]).

The energy associated with L is $E_L = \Delta(L) - L$.

The Euler-Lagrange equations corresponding to L can then be written in the form

$$X \,\lrcorner\, \omega_L = -\,\mathrm{d}E_L. \tag{2.2}$$

In coordinates we have the following expressions:

$$\theta_L = \frac{\partial L}{\partial v^j}\,\mathrm{d}q^j, \tag{2.3a}$$

$$\omega_L = \frac{\partial^2 L}{\partial v^i\,\partial v^j}\,\mathrm{d}v^i \wedge \mathrm{d}q^j + \frac{\partial^2 L}{\partial q^i\,\partial v^j}\,\mathrm{d}q^i \wedge \mathrm{d}q^j, \tag{2.3b}$$

$$\mathrm{d}E_L = \left(v^i\,\frac{\partial^2 L}{\partial v^i\,\partial q^j} - \frac{\partial L}{\partial q^j}\right)\mathrm{d}q^j + v^i\,\frac{\partial^2 L}{\partial v^i\,\partial v^j}\,\mathrm{d}v^j. \tag{2.3c}$$

If L is regular, i.e. if the Hessian matrix $(\partial^2 L/\partial v^i\,\partial v^j)$ is nonsingular, the 2-form ω_L is symplectic and the dynamical equation (2.2) possesses a unique solution X, the Euler-Lagrange vector field corresponding to L, which is moreover a second-order equation field. The latter can be easily verified by combining (2.1) and (2.2) and using the property

$$\Delta \,\lrcorner\, \omega_L = \mathrm{d}E_L \circ S, \tag{2.4}$$

(cf. [8]). For many interesting physical applications the regularity assumption is too restrictive and one has to extend the framework in order to allow for singular, i.e. degenerate, Lagrangians. Singularity of L in particular implies that ω_L is no longer of maximal rank. We will always assume, however, that the rank of ω_L is constant on TQ, which means that ω_L is a presymplectic form according to the following general definition:

DEFINITION 2.1. A presymplectic form on a manifold is a closed 2-form of constant rank. ■

It then follows that the set

$$\operatorname{char}\omega_L = \{\xi \in T(TQ) \mid \xi \,\lrcorner\, \omega_L(\tau_{Q*}(\xi)) = 0\}$$

is a subbundle of $T(TQ)$ which moreover defines an involutive distribution on TQ, the characteristic distribution of ω_L. With some abuse of notation we will also write $\operatorname{char}\omega_L$ for the set of smooth sections of this bundle, i.e. for the set of characteristic vector fields of ω_L. More precisely, a vector field Z on TQ

is said to belong to char ω_L if $Z \rfloor \omega_L = 0$.

For a degenerate Lagrangian, the dynamical equation $X \rfloor \omega_L = -\mathrm{d}E_L$ will not possess a globally defined solution in general, and even if it exists it will not be unique. The condition for the existence of a global solution is that E_L should be constant on the leaves of the foliation defined by char ω_L, i.e. for each characteristic vector field Z of ω_L one should have $\langle Z, \mathrm{d}E_L \rangle = 0$. If these equations are not identically satisfied they give rise to contraints. For the study of the solvability of the dynamical equation (2.2) one can then appeal to the geometric constraint algorithm developed by Gotay et al. [2], [5], [6].

In our analysis of the reduction problem for degenerate Lagrangians we will confine ourselves, however, to those Lagrangians for which the equation (2.2) possesses a global solution. For the sake of clarity we adopt here the following definition:

DEFINITION 2.2. A (degenerate) Lagrangian L is said to admit a global dynamics if there exists a globally defined vector field X on TQ satisfying $X \rfloor \omega_L = - \mathrm{d}E_L$. ■

The following result is straighforward.

PROPOSITION 2.1. *If L admits a global dynamics then the general solution of the dynamical equation (2.2) is of the form $X + Z$, where X is any particular solution and Z belongs to char ω_L.* ■

In order to study the reduction of degenerate Lagrangian systems, with ω_L presymplectic, we also have to require that the foliation defined by char ω_L is actually a fibration. This implies that the quotient or leaf space $TQ/\mathrm{char}\,\omega_L$ admits a manifold structure and the projection $\pi_L : TQ \to TQ/\mathrm{char}\,\omega_L$ becomes a surjective submersion.

Summarizing, we can say that the subsequent analysis of degenerate Lagrangians will rely on the following three basic assumptions:

(A1) ω_L is presymplectic;
(A2) L admits a global dynamics (in the sense of definition 2.2);
(A3) the foliation defined by char ω_L is a fibration.

2.2. A classification of Lagrangians

From (2.1) and the property $\mathrm{Im}\, S = V(TQ)$ it immediately follows that

$$S(\text{char}\ \omega_L) \subset \text{char}\ \omega_L \cap V(TQ) \tag{2.5}$$

for any (degenerate) Lagrangian L. For brevity we put

$$\text{char}\ \omega_L \cap V(TQ) = V(\text{char}\ \omega_L).$$

The same notation will be used for the set of vertical characteristic vector fields of ω_L. The following properties of $V(\text{char}\ \omega_L)$ are easily verified:

(2.6)
(i) $V(\text{char}\ \omega_L)$ defines an integrable distribution on TQ;
(ii) $V(\text{char}\ \omega_L) = \text{Ker}\,(S|_{\text{char}\ \omega_L})$.

A simple algebraic argument based on (2.5) and (2.6) reveals the following remarkable restriction on the dimension of the distribution $V(\text{char}\ \omega_L)$:

$$\dim\,[V(\text{char}\ \omega_L)] \geqslant \frac{1}{2} \dim\,[\text{char}\ \omega_L]$$

(see also [10]). Using this property we can now distinguish three types of Lagrangians:

Type I : if $\dim\,[\text{char}\ \omega_L] = \dim\,[V(\text{char}\ \omega_L)] = 0$;
Type II : if $\dim\,[\text{char}\ \omega_L] = 2\dim\,[V(\text{char}\ \omega_L)] \neq 0$;
Type III : if $\dim\,[\text{char}\ \omega_L] < 2\dim\,[V(\text{char}\ \omega_L)]$.

Type I Lagrangians are just regular Lagrangians, i.e. those for which ω_L is symplectic. As far as the reduction problem for degenerate Lagrangians is concerned it will be seen that type II Lagrangians and, in particular, a specific subclass of them, will play a prominent role. An immediate characterization of type II Lagrangians is given by:

LEMMA 2.1. *A Lagrangian L is of type II if and only if* $S(\text{char}\ \omega_L) = V(\text{char}\ \omega_L)$.

Proof. From (2.6) we deduce that $\dim\,[\text{char}\ \omega_L] - \dim\,[V(\text{char}\ \omega_L)] = \dim\,[S(\text{char}\ \omega_L)]$, from which the proof now easily follows, also taking into account (2.5). ∎

As mentioned already in the previous subsection, we will restrict our attention to Lagrangians which admit a global dynamics. From a physical point of view the cases of interest are mainly those for which among the possible solutions of the dynamical equation there exists a second-order equation field. In that respect the following theorem already underscores the importance of type II Lagrangians.

THEOREM 2.1. *If L is a type II Lagrangian which admits a global dynamics, then there exists a second-order equation field Γ on TQ satisfying $\Gamma \lrcorner \omega_L = - \mathrm{d}E_L$.*

Proof. Let X be any particular solution of the dynamical equation. Using (2.1), (2.2) and (2.4) we obtain

$$S(X) \lrcorner \omega_L = \mathrm{d}E_L \circ S$$
$$= \Delta \lrcorner \omega_L ,$$

from which it follows that $S(X) - \Delta \in \operatorname{char} \omega_L$. Since both $S(X)$ and Δ are vertical we even have $S(X) - \Delta \in V(\operatorname{char} \omega_L)$. According to lemma 2.1 there then exists a vector field $Z \in \operatorname{char} \omega_L$ such that

$$S(X) - \Delta = S(Z).$$

Clearly, $\Gamma = X - Z$ is also a solution of the dynamical equation (cf. proposition 2.1) and, moreover, the previous relation implies that $S(\Gamma) = \Delta$, i.e. Γ is a second-order equation field. ∎

This result has previously been established in [10] in the context of type II Lagrangians with constraints. However, in general, the second-order equation field constructed in that broader context need not be tangent to the constraint submanifold and so its relevance then becomes rather doubtful.

For a discussion of the second-order equation problem for general degenerate Lagrangians in the presence of constraints, we also refer to [6].

Before proceeding we first recall that for any vector field X on the base mani told Q one can detine the certical and the complete lift to TQ, denoted by X^v and X^c, respectively (see e.g. [8]). In coordinates, putting $X = \lambda^i \, \partial/\partial q^i$, we have

$$X^v = \lambda^i \frac{\partial}{\partial v^i} , \quad X^c = \lambda^i \frac{\partial}{\partial q^i} + v^j \frac{\partial \lambda^i}{\partial q^j} \frac{\partial}{\partial v^i} .$$

For any two vector fields X and Y on Q one can verify that

$$[X^c, Y^c] = [X, Y]^c, \quad [X^v, Y^c] = [X, Y]^v, \quad [X^v, Y^v] = 0, \tag{2.7}$$

and

$$\mathcal{L}_{X^c} S = \mathcal{L}_{X^v} S = 0 \tag{2.8}$$

(cf. [8]).

If for a given degenerate Lagrangian L and some vector field X on Q we have $X^c \in \operatorname{char} \omega_L$, then by (2.5) $X^v = S(X^c) \in V(\operatorname{char} \omega_L)$. Moreover, using the first relation of (2.7) and taking account of the integrability of $\operatorname{char} \omega_L$, the following result is straightforward.

PROPOSITION 2.2. *The set* $\mathcal{D} = \{X \in \mathcal{X}(Q) \mid X^c \in \operatorname{char} \omega_L\}$ *defines an integrable distribution on* Q. ∎

Let g be a Lie algebra of vector fields on Q. We call the tangent algebra of g, denoted by Tg, the algebra defined by the complete and vertical lifts of vector fields in g. Each element of Tg can be identified with a couple (X^c, Y^v) for some $X, Y \in g$. Taking account of the bracket relations (2.7) one can then easily establish an isomorphism between Tg and the semi-direct product $g \circledcirc g$ with the Lie bracket on $g \oplus g$ defined in terms of the adjoint action of g onto itself, i.e. $[(X_1, Y_1), (X_2, Y_2)] = ([X_1, X_2], \operatorname{ad} X_1(Y_2) - \operatorname{ad} X_2(Y_1))$, with $\operatorname{ad} X(Y) = [X, Y]$. The concept of a tangent algebra of vector fields in fact only requires an integrable distribution on Q, i.e. g need not be a finite dimensional Lie algebra. Let us first consider an arbitrary distribution $\mathcal{D}$ on Q. The span of the set of all vertical and complete lifts of vector fields in $\mathcal{D}$ is a distribution on TQ which we call the «tangent distribution» of $\mathcal{D}$. The fact that this procedure actually defines a distribution on TQ may not be immediately obvious in view of the non-local nature of the complete lift.However, for any vector field X and any function f on Q we have

$$(fX)^c = (\tau_Q^* f) X^c + (\widehat{\mathrm{d}f}) X^v,$$

where $\widehat{\mathrm{d}f}$ is the function on TQ defined by $\widehat{\mathrm{d}f}(x) = \langle x, \mathrm{d}f(\tau_Q(x))\rangle$. So, $(fX)^c$ lies in the span of X^c and X^v. It follows that if $X_1, \ldots, X_k$ is a local basis for $\mathcal{D}$ then each vertical and complete lift of a vector field in $\mathcal{D}$ lies in the span of $X_1^c, X_2^c, \ldots, X_k^c, X_1^v, X_2^v, \ldots, X_k^v$. Thus the tangent distribution of $\mathcal{D}$ is a distribution of dimension $2\dim(\mathcal{D})$. Its vertical part, which is always integrable, is of dimension $\dim(\mathcal{D})$ and the tangent distribution as a whole is integrable if and only if $\mathcal{D}$ is integrable. In that case the tangent distribution is indeed a tangent algebra in the sense defined above.

Given a degenerate Lagrangian L on TQ we introduce the following definition.

DEFINITION 2.3. $\operatorname{char} \omega_L$ is a tangent distribution if there is a distribution $\mathcal{D}$ on Q such that $\operatorname{char} \omega_L$ is spanned by the complete and vertical lifts of vector fields in $\mathcal{D}$. ∎

From the previous considerations it follows that the distribution $\mathcal{D}$ is necessarily integrable.

If $\operatorname{char} \omega_L$ is a tangent distribution it admits a local basis of the form $X_1^c, X_2^c, \ldots, X_k^c, X_1^v, X_2^v, \ldots, X_k^v$, for some vector fields X_i on Q. Since $S(X_i^c) = X_i^v$ one can easily see that $S(\mathrm{char}\,\omega_L) = V(\mathrm{char}\,\omega_L)$. Consequently, a Lagrangian for which $\mathrm{char}\,\omega_L$ is a tangent distribution is necessarily of type II.

2.3. Lagrangian systems on integrable almost tangent manifolds

Suppose L is a Lagrangian defined on TQ such that the assumptions (A1), (A2) and (A3) from section 2.1 hold. The idea of reduction we wish to develop further on consists, roughly speaking, of finding conditions under which one can associate to the given degenerate system a regular Lagrangian system on the quotient space $TQ/\mathrm{char}\,\omega_L$. An apparent prerequisite for such a reduction is that it should be possible to conceive the quotient space as a tangent bundle of some manifold or, at least, to provide it with an integrable almost tangent structure. In a previous paper by one of us it has indeed been pointed out that the framework of integrable almost tangent structures lends itself already to the conception of a Lagrangian dynamics (cf. [11]). The purpose of this section is to elaborate a bit further on this idea.

First, recall that an integrable almost tangent structure on a manifold M is defined by a type (1,1) tensor field $\tilde{S}$ such that for each $m \in M$ the linear endomorphism $\tilde{S}_m$ satisfies $\mathrm{Im}\,(\tilde{S}_m) = \mathrm{Ker}\,(\tilde{S}_m)$, and that the Nijenhuis tensor of $\tilde{S}$ vanishes. The latter in particular entails that the distribution defined by $\mathrm{Im}\,(\tilde{S})$ on M is involutive. Clearly, M must be even dimensional and, moreover, one can always locally find adapted coordinates (q, v) in terms of which $\tilde{S} = \partial/\partial v^i \otimes \mathrm{d}q^i$. If the foliation induced by $\mathrm{Im}\,(\tilde{S})$ on M is a fibration and if the leaves satisfy a certain completeness condition then, by a suitable choice of a zero section, M becomes isomorphic to a tangent bundle. For more details, see e.g. [11], [14], [15] and references therein. On an integrable almost tangent manifold $(M, \tilde{S})$ we consider the class of vector fields Γ for which

$$\mathcal{L}_\Gamma \tilde{S} \circ \tilde{S} = \tilde{S}, \quad \tilde{S} \circ \mathcal{L}_\Gamma \tilde{S} = -\tilde{S}, \tag{2.9}$$

(the tensor fields are regarded here as operators on vector fields). These properties are in particular shared by any second-order equation field on a tangent bundle, with respect to the canonical type (1,1) tensor field S (see e.g. [9]). Moreover, it has recently been argued in Sarlet et al. [13] that it are precisely the relations (2.9) which play an important role in the study of some properties of Lagrangian systems, rather than the more stringent condition $S(\Gamma) = \Delta$. In adapted coordinates, a vector field Γ satisfying (2.9) is of the form $\Gamma = (v^i + \lambda^i(q))\,\partial/\partial q^i + \Lambda^i(q, v)\,\partial/\partial v^i$.

Inspired by the necessary and sufficient conditions for a second-order equation field on a tangent bundle to be (locally) an Euler-Lagrange vector field (see e.g. [12]), we propose the following notion of regular Lagrangian system on an integrable almost tangent manifold $(M, \tilde{S})$.

DEFINITION 2.4. A vector field Γ, satisfying (2.9), is called a regular Lagrangian system if there exists a symplectic form ω on M such that

(i) for each $m \in M$ the subspace $\operatorname{Im}(\tilde{S}_m)$ of $T_m M$ is Lagrangian with respect to ω_m, i.e., $\omega_m(x, y) = 0$ for all $x, y \in \operatorname{Im}(\tilde{S}_m)$;

(ii) $\mathcal{L}_\Gamma \omega = 0$. ■

Using (i) and (ii) one can show that $\tilde{S}$ and ω are related by

$$\tilde{S}(X) \lrcorner \omega = -(X \lrcorner \omega) \circ \tilde{S}, \tag{2.10}$$

Moreover, if Γ is a regular Lagrangian system in the sense of the previous definition, then there locally exists a function L such that

$$\omega = \mathrm{d}(\mathrm{d}L \circ \tilde{S}) \quad \text{and} \quad \Gamma \lrcorner \omega = -\mathrm{d}E_L, \tag{2.11}$$

with $E_L = \tilde{S}(\Gamma)(L) - L$. The proofs of (2.10) and (2.11) essentially proceed along the same lines as those given for the corresponding properties in the framework of ordinary Lagrangian mechanics in [8] and [12], respectively.

2.4. Reduction of degenerate Lagrangian systems

Let L be a degenerate Lagrangian on TQ for which the assumptions (A1), (A2) and (A3) are satisfied. Suppose furthermore that the dynamical equation (2.2) admits a second-order equation solution Γ. We now wish to investigate under what conditions the quotient space $TQ/\operatorname{char}\omega_L$ can be equipped with an integrable almost tangent structure and a suitable symplectic structure such that Γ projects onto a regular Lagrangian system with respect to these structures, in the sense of definition 2.4.

An immediate observation is that under the given assumptions there exists a unique symplectic form $\tilde{\omega}$ on $TQ/\operatorname{char}\omega_L$ such that

$$\omega_L = \pi_L^* \tilde{\omega} \tag{2.12}$$

where $\pi_L : TQ \to TQ/\operatorname{char}\omega_L$ is the projection mapping. This is in fact a general result for the reduction of presymplectic structures (see e.g. [16], theorem 25.2).

The next step in the reduction process consists of finding conditions under which the canonical type (1,1) tensor field S on TQ passes to the quotient such that its projection $\tilde{S}$ defines an integrable almost tangent structure on $TQ/\operatorname{char}\omega_L$ and such that the spaces $\operatorname{Im}(\tilde{S}_m)$ are Lagrangian with respect to $\tilde{\omega}_m$. Criteria for a general type (1,1) tensor field to be projectable onto the leaf space of an integrable distribution, are given by:

PROPOSITION 2.3. *Let $\mathcal{D}$ be an integrable distribution on a manifold M such that the foliation defined by $\mathcal{D}$ is a fibration. A type (1,1) tensor field R on M projects* *onto the quotient space $M/\mathcal{D}$ iff*

(i) $R(\mathcal{D}) \subset \mathcal{D}$

(ii) $\mathrm{Im}\,(\mathcal{L}_Z R) \subset \mathcal{D}$ *for each* $Z \in \mathcal{D}$.

Proof. See Appendix A. ■

In the case we are dealing with here we already known that $S(\mathrm{char}\,\omega_L) \subset \subset \mathrm{char}\,\omega_L$, i.e. condition (i) of the previous proposition is satisfied. Hence, we immediately obtain

COROLLARY. *S passes to the quotient under* char ω_L *iff*

$$\mathrm{Im}\,(\mathcal{L}_Z S) \subset \mathrm{char}\,\omega_L \quad \textit{for each} \quad Z \in \mathrm{char}\,\omega_L . \tag{2.13}$$ ■

We now arrive at the following important result (with $\tilde{\omega}$ always denoting the symplectic form on $TQ/\mathrm{char}\,\omega_L$ for which (2.12) holds).

PROPOSITION 2.4. *If S passes to the quotient under* char ω_L *then the projected tensor field $\tilde{S}$ defines an integrable almost tangent structure if and only if L is of type II. Moreover, in that case the subspaces* $\mathrm{Im}\,(\tilde{S}_m)$ *are Lagrangian with respect to $\tilde{\omega}_m$ for each $m \in TQ/$*char ω_L.

Proof. If S is projectable, with projection $\tilde{S}$, then $N_S = 0$ implies $N_{\tilde{S}} = 0$.

This follows from the fact that if a type (1,1) tensor field is projectable under a surjective submersion π, then the Nijenhuis tensors of the given and of the projected tensor field are π-related (see e.g. [17]). Clearly, $S^2 = 0$ also implies $\tilde{S}^2 = 0$ and so we already have $\mathrm{Im}\,(\tilde{S}_m) \subset \mathrm{Ker}\,(\tilde{S}_m)$ for each $m \in TQ/\mathrm{char}\,\omega_L$. For $\tilde{S}$ to be an integrable almost tangent structure the missing bit is the opposite inclusion, namely $\mathrm{Ker}\,(\tilde{S}_m) \subset \mathrm{Im}\,(\tilde{S}_m)$.

First, suppose L is of type II. Let $\tilde{\xi} \in \mathrm{Ker}\,(\tilde{S}_m)$ and take $y \in \pi_L^{-1}\{m\}$ and $\xi \in T_y(TQ)$ such that $(\pi_L)_*(\xi) = \tilde{\xi}$. Using the definition of $\tilde{S}$ we find that $(\pi_L)_*(S_y(\xi)) = \tilde{S}_m(\tilde{\xi}) = 0$, whence $S_y(\xi) \in V_y(\mathrm{char}\,\omega_L)$. From lemma 2.1 it then follows that $\xi = \eta + S_y(\zeta)$ for some $\eta \in \mathrm{char}\,\omega_L$ and $\zeta \in T_y(TQ)$. Consequently, $\tilde{\xi} = (\pi_L)_*(S_y(\zeta)) = \tilde{S}_m(\tilde{\zeta})$, with $\tilde{\zeta} = (\pi_L)_*(\zeta)$, and thus $\tilde{\xi} \in \mathrm{Im}\,(\tilde{S}_m)$.

Conversely, suppose $\mathrm{Ker}\,(\tilde{S}_m) \subset \mathrm{Im}\,(\tilde{S}_m)$ for each m. We shall then prove that $V(\mathrm{char}\,\omega_L) \subset S(\mathrm{char}\,\omega_L)$ which, in view of (2.5) and lemma 2.1, yields that L is of type II.

Take $\xi \in V_y(\mathrm{char}\,\omega_L)$. We then have that $\xi = S_y(\eta)$ for some $\eta \in T_y(TQ)$. Putting $m = \pi_L(y)$ and $\tilde{\eta} = (\pi_L)_*(\eta)$ one can easily see that $\tilde{\eta} \in \mathrm{Ker}\,(\tilde{S}_m)$ and thus, by assumption, $\tilde{\eta} \in \mathrm{Im}\,(\tilde{S}_m)$. Herewith, it is straightforward to verify that there exists a $\eta_1 \in \operatorname{char} \omega_L \cap T_y(TQ)$ such that $S_y(\eta_1) = S_y(\eta) = \xi$ and, hence, $\xi \in S_y(\operatorname{char} \omega_L)$. This completes the proof of the first part of the theorem.

In order to prove the second assertion, let $\tilde{\xi} \in T_m(TQ/\operatorname{char} \omega_L)$ and take again $y \in \pi_L^{-1}\{m\}$ and $\xi \in T_y(TQ)$ such that $(\pi_L)_*(\xi) = \tilde{\xi}$. The following relations are easily verified

$$S_y(\xi) \lrcorner (\omega_L)_y = (\tilde{S}_m(\tilde{\xi}) \lrcorner \tilde{\omega}_m) \circ (\pi_L)_*$$

and

$$(\xi \lrcorner (\omega_L)_y) \circ S_y = (\tilde{\xi} \lrcorner \tilde{\omega}_m) \circ \tilde{S}_m \circ (\pi_L)_*,$$

which, by virtue of (2.1) and the fact that π_L is a surjective submersion, yield

$$-(\tilde{\xi} \lrcorner \tilde{\omega}_m) \circ \tilde{S}_m = \tilde{S}_m(\tilde{\xi}) \lrcorner \tilde{\omega}_m.$$

Taking into account that, $\tilde{S}$ being an integrable almost tangent structure, we already have that $\dim [\operatorname{Im} (\tilde{S}_m)] = \frac{1}{2} \dim [T_m(TQ/\operatorname{char} \omega_L)]$, the previous relation indeed implies that $\operatorname{Im} (\tilde{S}_m)$ is a Lagrangian subspace of $T_m(TQ/\operatorname{char} \omega_L)$. ■

The above proposition already limits the class of Lagrangians for which the proposed reduction scheme might work to those Lagrangians which are of type II. Before continuing the reduction analysis for type II Lagrangians we first mention a few general properties, always assuming we are dealing with a degenerate Lagrangian for which (A1), (A2) and (A3) hold.

LEMMA 2.2. *Each vector field X satisfying $X \lrcorner \omega_L = -\mathrm{d}E_L$ is projectable under* $\operatorname{char} \omega_L$. ■

LEMMA 2.3. *The energy function E_L is projectable under* $\operatorname{char} \omega_L$. ■

The proofs of these two lemmas are straightforward. Lemma 2.2 is in fact a particular case of the following more general result: if X is any locally Hamiltonian vector field with respect to ω_L, i.e. $\mathcal{L}_X \omega_L = 0$, then $[X, Z] \in \operatorname{char} \omega_L$ for each $Z \in \operatorname{char} \omega_L$ and, hence, X is projectable.

Finally, let us have a look at the projectability condition for the Poincaré-Cartan form θ_L. By definition of $\operatorname{char} \omega_L$ we already know that for each $Z \in \operatorname{char} \omega_L$, $Z \lrcorner \mathrm{d}\theta_L = 0$. The necessary and sufficient condition for projectability of θ_L therefore is that for each $Z \in \operatorname{char} \omega_L$

$$\langle Z, \theta_L \rangle = 0,$$

or, equivalently, since $\theta_L = \mathrm{d}L \circ S$,

$$S(Z)(L) = 0. \tag{2.14}$$

Clearly, projectability of θ_L need not imply projectability of L. However, the converse is true. Indeed, if L is projectable then in particular, taking into account (2.5), (2.14) holds and so θ_L is projectable. If both S and L are projectable one immediately obtains that θ_L projects onto $\tilde{\theta}_L = \mathrm{d}\tilde{L} \circ \tilde{S}$.

We now turn our attention to Lagrangians of type II.

2.5. Reduction of type II Lagrangians

Throughout this subsection we will be dealing with Lagrangians of type II for which it will always be assumed that (A1), (A2) and (A3) hold. Before discussing the reduction problem we first mention the following interesting property.

PROPOSITION 2.5. *If L is a Lagrangian of type II then the leaves of* char ω_L *possess an integrable almost tangent structure.*

Proof. Since, according to (2.5), S leaves char ω_L invariant, one can consider the restriction $\bar{S}$ of S to each leaf. The properties $\operatorname{Ker}(\bar{S}) = \operatorname{Im}(\bar{S})$ and $N_{\bar{S}} = 0$ can then easily be deduced from the corresponding properties of S, taking account of the fact that for type II Lagrangians $S(\text{char}\,\omega_L) = V(\text{char}\,\omega_L)$ (cf. lemma 2.1). Consequently, the restriction of S defines an integrable almost tangent structure on each leaf. ■

We have already seen that a special class of type II Lagrangians consists of those for which char ω_L is a tangent distribution (cf. definition 2.2). It turns out that for such Lagrangians the tensor field S is always projectable under char ω_L.

PROPOSITION 2.6. *If* char ω_L *is a tangent distribution, then S projects onto* $TQ/\text{char}\,\omega_L$.

Proof. According to the corollary of proposition 2.3 we only need to prove (2.13), i.e. $\operatorname{Im}(\mathcal{L}_Z S) \subset \text{char}\,\omega_L$ for each $Z \in \text{char}\,\omega_L$. Since char ω_L is a tangent distribution it admits a local basis of the form $X_1^c, \ldots, X_k^c, X_1^v, \ldots, X_k^v$ for some vector fields X_i on Q, and so it suffices to prove (2.13) for vector fields of the form fX_i^c and fX_i^v with f a smooth function on TQ.

First, observe that for any function f and any vector field Z on TQ we have

$$\mathcal{L}_{fZ} S = f\mathcal{L}_Z S + S(Z) \otimes \mathrm{d}f - Z \otimes (\mathrm{d}f \circ S),$$

as can be easily verified. Applying this to the case where $Z = X_i^c$ and $Z = X_i^v$ for some generating vector field X_i of char ω_L and taking account of (2.8) we find, respectively

$$\mathcal{L}_{fX_i^c} S = X_i^v \otimes \mathrm{d}f - X_i^c \otimes (\mathrm{d}f \circ S)$$

and

$$\mathcal{L}_{fX_i^v} S = -X_i^v \otimes (\mathrm{d}f \circ S).$$

Since both X_i^c and X_i^v belong to char ω_L it follows that $\mathrm{Im}\,(\mathcal{L}_{fX_i^c} S) \subset$ char ω_L and $\mathrm{Im}\,(\mathcal{L}_{fX_i^v} S) \subset$ char ω_L. ∎

It can in fact be shown that the only type II Lagrangians for which S is projectable are those for which char ω_L is a tangent distribution. This is a consequence of the following general theorem.

THEOREM 2.2. *Let $\hat{\mathcal{D}}$ be a distribution on TQ such that*

(i) $\dim\,[V(TQ) \cap \hat{\mathcal{D}}] = \frac{1}{2} \dim \hat{\mathcal{D}}$,

(ii) $S(\hat{\mathcal{D}}) \subset \hat{\mathcal{D}}$

(iii) $\mathrm{Im}\,(\mathcal{L}_Z S) \subset \hat{\mathcal{D}}$ *for all* $Z \in V(TQ) \cap \hat{\mathcal{D}}$,

(iv) *there is a second-order equation field* Γ *on* TQ *such that* $[\Gamma, Z] \in \hat{\mathcal{D}}$ *for all* $Z \in \hat{\mathcal{D}}$.

Then there is a distribution $\mathcal{D}$ on Q of which $\hat{\mathcal{D}}$ is the tangent distribution. If $\hat{\mathcal{D}}$ is integrable, so is $\mathcal{D}$.

Proof. See Appendix B. ∎

From theorem 2.1 we known that a type II Lagrangian which admits a global dynamics will have a second-order equation solution. For such a Lagrangian all conditions of theorem 2.2 are verified with $\hat{\mathcal{D}}$ = char ω_L, except condition (iii). If the latter is also satisfied and thus, in particular, if S is projectable, then the theorem tells us that char ω_L is a tangent distribution. Combining this with the argument of proposition 2.4 it follows that the only possible candidates for admitting a regular Lagrangian reduction, in the sense described in the beginning of the previous section, are those Lagrangians for which char ω_L is a tangent distribution. The next theorem now states that for such Lagrangians the proposed reduction scheme effectively works.

THEOREM 2.3. *Let L be a Lagrangian for which* char ω_L *is a tangent distribution* *and such that (A1), (A2) and (A3) hold. Then, L induces a regular Lagrangian system on the quotient space* $TQ/\mathrm{char}\,\omega_L$.

Proof. Combining the results from propositions 2.4 and 2.6 we already know that $TQ/\mathrm{char}\,\omega_L$ inherits an integrable almost tangent structure $\tilde{S}$ and a symplectic structure $\tilde{\omega}$ with the property that $\mathrm{Im}\,(\tilde{S})$ defines a Lagrangian distribution with respect to $\tilde{\omega}$.

Now, let Γ be a second-order equation solution of the dynamical equation (2.2) corresponding to L (and, by virtue of theorem 2.1, such a Γ certainly exists). It follows from lemma 2.2 that Γ is projectable. The projected vector field which, as a matter of fact, is the same for any solution of (2.2), is denoted by $\tilde{\Gamma}$. Taking into account lemma 2.3 we find that

$$\tilde{\Gamma} \lrcorner\, \tilde{\omega} = -\,\mathrm{d}\tilde{E}_L ,$$

with $\tilde{E}_L$ the projection of E_L. In particular this implies that $\mathcal{L}_{\tilde{\Gamma}}\tilde{\omega} = 0$. According to definition 2.4 it only remains to be verified that $\tilde{\Gamma}$ satisfies the conditions (2.9), i.e., $\mathcal{L}_{\tilde{\Gamma}}\tilde{S} \circ \tilde{S} = \tilde{S}$ and $\tilde{S} \circ \mathcal{L}_{\tilde{\Gamma}}\tilde{S} = -\tilde{S}$. For that purpose we first prove that the tensor field $\mathcal{L}_\Gamma S$ passes to the quotient under $\mathrm{char}\,\omega_L$.

In view of (2.5) and the projectability of Γ, and using the identity $(\mathcal{L}_\Gamma S)(Z) = [\Gamma, S(Z)] - S([\Gamma, Z])$ it immediately follows that $\mathcal{L}_\Gamma S(\mathrm{char}\,\omega_L) \subset \mathrm{char}\,\omega_L$. Next, for any $Z \in \mathrm{char}\,\omega_L$ and any vector field X on TQ we have

$$[\mathcal{L}_Z(\mathcal{L}_\Gamma S)](X) = (\mathcal{L}_{[Z,\Gamma]}S)(X) - \mathcal{L}_\Gamma[\mathcal{L}_Z S(X)] + \mathcal{L}_Z S([\Gamma, X]).$$

Taking into account (2.13) and the fact that Γ is projectable, if is seen that each term on the right-hand side belongs to $\mathrm{char}\,\omega_L$. Consequently, $\mathrm{Im}\,[\mathcal{L}_Z(\mathcal{L}_\Gamma S)] \subset \subset \mathrm{char}\,\omega_L$ for each $Z \in \mathrm{char}\,\omega_L$ and so the conditions of proposition 2.3 are verified with $R = \mathcal{L}_\Gamma S$ and $\mathcal{D} = \mathrm{char}\,\omega_L$. The tensor field $\mathcal{L}_\Gamma S$ is thus projectable and, clearly, its projection is precisely $\mathcal{L}_{\tilde{\Gamma}}\tilde{S}$. The relations (2.9) are now an immediate consequence of the corresponding properties of the second-order equation field Γ (cf. section 2.3). This completes the proof of the theorem. ■

We may thus conclude that Lagrangians for which $\mathrm{char}\,\omega_L$ is a tangent distribution admit a regular Lagrangian reduction and, moreover, according to the discussion preceding theorem 2.3, they are also the only ones having this property.

Remark. Under the conditions of theorem 2.3 we also have that the dilation field Δ is projectable. Indeed, since $\Delta = S(\Gamma)$ we find that for each $Z \in \mathrm{char}\,\omega_L$

$$[Z, \Delta] = (\mathcal{L}_Z S)(\Gamma) + S([Z, \Gamma]).$$

In view of (2.5), (2.13) and the fact that Γ is projectable, it is clear that $[Z, \Delta] \in \in \operatorname{char} \omega_L$ for each $Z \in \operatorname{char} \omega_L$ and, hence, Δ is projectable. Its projection is $\tilde{\Delta} = \tilde{S}(\tilde{\Gamma})$.

Suppose again the conditions of theorem 2.3 are satisfied. Let $\tilde{L}_1$ be a (local) Lagrangian for the projected vector field $\tilde{\Gamma}$, in the sense described at the end of section 2.3, i.e. $\tilde{\omega} = \mathrm{d}(\mathrm{d}\tilde{L}_1 \circ \tilde{S})$ and $\tilde{\Gamma} \lrcorner \tilde{\omega} = -\mathrm{d}E_{\tilde{L}_1}$ where $E_{\tilde{L}_1} = \tilde{S}(\tilde{\Gamma})(\tilde{L}_1) - -\tilde{L}_1$ which, in view of the previous remark, can be rewritten as $E_{\tilde{L}_1} = \tilde{\Delta}(\tilde{L}_1) - -\tilde{L}_1$. Putting $L_1 = \pi_L^* \tilde{L}_1$, the next proposition extablishes the relationship between the (local) «lifted» Lagrangian L_1 and the original Lagrangian L.

First we notice that although strictly speaking the function $\tilde{L}_1$ need not be globally defined on $TQ/\operatorname{char} \omega_L$, we will systematically ignore, for notational convenience, any indication of restrictions on the domains of functions and forms. This should not give rise to any confusion, however.

We also recall that any 1-form α on Q can be regarded in a natural way as a function on TQ, henceforth denoted by $\hat{\alpha}$, with $\hat{\alpha}(x) = \langle x, \alpha(\tau_Q(x))\rangle$ for $x \in TQ$.

PROPOSITION 2.7. *L_1 and L are gauge equivalent, i.e., there exists a closed 1-form α on Q such that $L = L_1 + \hat{\alpha}$ (up to a constant).*

Proof. In the proof of proposition 2.5 we have already observed that $\tilde{\Gamma} \lrcorner \tilde{\omega} = = -\mathrm{d}\tilde{E}_L$ with $\tilde{E}_L$ the projection of E_L. On the other hand we also have $\tilde{\Gamma} \lrcorner \tilde{\omega} = = -\mathrm{d}E_{\tilde{L}_1}$ from which

$$\mathrm{d}\tilde{E}_L = \mathrm{d}E_{\tilde{L}_1}. \tag{2.15}$$

Furthermore,

$$\omega_L = \pi_L^* \tilde{\omega} = \pi_L^*[\mathrm{d}(\mathrm{d}\tilde{L}_1 \circ \tilde{S})] = \omega_{L_1}$$

and,

$$\pi_L^* E_{\tilde{L}_1} = \pi_L^*(\tilde{\Delta}(\tilde{L}_1) - \tilde{L}_1) = \Delta(L_1) - L_1 = E_{L_1},$$

which, combined with (2.15), yields $\mathrm{d}E_{L_1} = \mathrm{d}E_L$. This means that, in the terminology of [10], the Lagrangians L and L_1 are equivalent (i.e. they determine the same dynamics) and geometrically equivalent (i.e. $\omega_L = \omega_{L_1}$). Since by assumption there exists a second-order equation field Γ for which $\Gamma \lrcorner \omega_L = -\mathrm{d}E_L$, it follows from [10] (theorem 2) that L and L_1 are gauge equivalent. ∎

Summarizing we can say that under the present assumptions L is always locally gauge equivalent to a Lagrangian which projects onto a regular Lagrangian for the reduced system. Notice, however, that L itself need not be projectable. As far as the projectability of L is concerned we have the following interesting property.

PROPOSITION 2.8. *Let L be a Lagrangian which, in particular, is smooth on the zero section of TQ and for which (A1), (A2) and (A3) hold. Suppose* char ω_L *is a tangent distribution, then θ_L is projectable if and only L is projectable.*

Proof. The fact that projectability of L implies projectability of θ_L has already been observed at the end of the previous subsection.

Suppose now that θ_L is projectable such that

$$\langle Z, \theta_L \rangle = 0, \quad \text{for all} \quad Z \in \text{char}\ \omega_L.$$

Since char ω_L is a tangent distribution it suffices to prove $X^c(L) = 0$ and $X^v(L) = = 0$ for any vector field X on Q whose complete and vertical lifts belong to char ω_L. Since $X^v = S(X^c)$ we already know that projectability of θ_L implies $X^v(L) = 0$ (see (2.14)). From lemma 2.3 we deduce that $X^c(E_L) = 0$ for $X^c \in \in$ char ω_L. By definition of E_L this implies

$$X^c(\Delta(L)) = X^c(L)$$

or, since $[X^c, \Delta] = 0$ for each vector field X on Q (cf. [8]),

$$\Delta(X^c(L)) = X^c(L),$$

which means that $X^c(L)$ is homogeneous of degree 1 in the velocities. Since L is smooth on the zero section of TQ, this in turn implies that $X^c(L)$ must be linear in the velocities, i.e.

$$X^c(L) = \hat{\zeta}$$

for some 1-form ζ on Q. On the other hand, for $Z \in$ char ω_L and $\langle Z, \theta_L \rangle = 0$ we have that $\mathcal{L}_Z \theta_L = 0$, or

$$\mathrm{d}(Z(L)) \circ S + \mathrm{d}L \circ \mathcal{L}_Z S = 0.$$

Putting $Z = X^c$ and taking account of (2.8) we find

$$\mathrm{d}\hat{\zeta} \circ S = \tau_Q^* \zeta = 0.$$

Hence, $\zeta = 0$ and thus $X^c(L) = 0$, which completes the proof. ∎

If L is projectable then its projection of course yields a regular Lagrangian for the reduced system.

We finally make the following observation. Let L be a Lagrangian for which char ω_L is the tangent distribution of a distribution $\mathcal{D}$ on Q. From proposition 2.2 we know that $\mathcal{D}$ is necessarily integrable. Suppose the foliation defined by $\mathcal{D}$ is a fibration such that the leaf space $Q/\mathcal{D}$ admits a manifold structure and the projection $\pi : Q \to Q/\mathcal{D}$ is a surjective submersion. It is then easily seen that $TQ/\text{char } \omega_L$ can be canonically identified with $T(Q/\mathcal{D})$ and $\pi_L = \pi_*$. The projection of S under char ω_L then coincides with the standard almost tangent structure on $T(Q/\mathcal{D})$ and the reduced regular Lagrangian system is a genuine Euler-Lagrange vector field on a tangent bundle. This situation occurs for instance when there is a free and proper action of a Lie group G on Q such that char ω_L is spanned by the complete and vertical lifts of the infinetesimal generators of this action. We will meet this case in the next section.

3. DEGENERATE LAGRANGIANS AND THE REDUCTION OF HAMILTONIAN SYSTEMS WITH SYMMETRIES

The aim of this section is to study the relation between the Dirac theory of constraints and the reduction theory for degenerate Lagrangian systems as described in the previous section. In particular we will reveal a deep link between both points of view in a very nice situation, well-known from the study of the reduction of dynamical systems with symmetry and the momentum map (see e.g. [3], [18], [19]).

3.1. Dirac's theory of constraints and the Legendre map

Since the pioneering work by Dirac [1] and Bergmann [20] on constrained systems, much has been written on the subject and there is certainly no point in trying to give a complete survey of the relevant literature. For a comprehensive review of the theory and further references, the reader may consult for instance Hanson et al. [21] and Sundermeyer [22], whereas we also would like to mention the nice geometrical treatment on the dynamics and symmetries of constrained systems by Marmo et al. [23]. A general geometrical setting for the analysis of constrained systems, which globalizes Dirac's local treatment, has been accomplished by Gotay et al. [2], [5], [6]. They have developed a geometric constraint algorithm which applies to Hamiltonian systems on general presymplectic manifolds. We will briefly sketch an adapted version of this algorithm in the canonical picture.

Suppose we are given a Lagrangian L on TQ for which ω_L is presymplectic. The associated Legendre map will be denoted by FL, i.e. $FL : TQ \to T^*Q$ is the fiber derivative of L which in natural bundle coordinates reads

$$FL(q, v) = \left(q, \frac{\partial L}{\partial v}\right),$$

(with obvious shorthand notation). L is regular iff FL is a local diffeomorphism and L is called hyperregular if FL is a global diffeomorphism (see e.g. [19]). If θ_Q represents the canonical (or Liouville) 1-form on T^*Q and $\Omega_Q = \mathrm{d}\theta_Q$ the canonical symplectic form, then it can be shown that $FL^*\theta_Q = \theta_L$ and thus also

$$FL^*\Omega_Q = \omega_L$$

(see e.g. [5]). In fact, this is sometimes used as a definition for ω_L. If L is degenerate then $\mathrm{Im}\,(FL)$ will in general be a submanifold of T^*Q on which Ω_Q induces, by restriction, a presymplectic structure. An important observation is that

$$V(\mathrm{char}\,\omega_L) = \mathrm{Ker}\,(FL_*). \tag{3.1}$$

This can be easily verified using coordinate expressions, for instance. For brevity we henceforth put

$$\mathrm{Im}\,(FL) = M_1, \qquad \Omega_Q|_{M_1} = \Omega_1.$$

M_1 is called the primary constraint submanifold and any function on T^*Q which is constant on M_1 is called a primary constraint.

For the subsequent development we need the notion of almost regularity of a Lagrangian, which was introduced in [5].

DEFINITION 3.1. A Lagrangian L is called almost regular if the Legendre map FL is a submersion onto its image and the fibres $FL^{-1}\{FL(x)\}$ are connected for each $x \in TQ$. ∎

For an almost regular Lagrangian it follows from (3.1) that M_1 can be identified with (i.e., is canonically diffeomorphic to) the leaf space of the integrable distribution defined by $V(\mathrm{char}\,\omega_L)$ on TQ. Moreover, it can then be shown that the energy function E_L is FL-projectable, i.e., there exists a smooth function H_1 on M_1 such that

$$H_1 \circ FL = E_L \tag{3.2}$$

(cf. [5]). In this way the given almost regular Lagrangian L induces a Hamiltonian system on the presymplectic manifold (M_1, Ω_1) with dynamical equation

$$Y \,\lrcorner\, \Omega_1 = -\,\mathrm{d}H_1. \tag{3.3}$$

The solvability of this equation can then be studied by means of the geometric constraint algorithm, which locally corresponds to the Dirac theory of constraints. The condition for there to exist a solution of (3.3) on M_1 is that at each point $m \in M_1$

(3.4) $$\langle \xi, \mathrm{d}H_1(m)\rangle = 0, \quad \text{for all} \quad \xi \in T_m M_1^{\S},$$

where $T_m M_1^{\S}$ is the Ω_1-orthogonal complement of $T_m M_1$, i.e., $T_m M_1^{\S} = \{\xi \in \in T_m M_1 \mid (\Omega_1)_m(\xi, \eta) = 0 \text{ for all } \eta \in T_m M_1\}$.

Note that $\bigcup_{m \in M_1} T_m M_1^{\S} = \operatorname{char} \Omega_1$, the characteristic distribution of Ω_1.

Let $j_1 : M_1 \to T^*Q$ denote the natural inclusion and let $FL_1 : TQ \to M_1$ be the map induced by FL such that the diagram

$$\begin{tikzcd} TQ \arrow[r, "FL"] \arrow[dr, "FL_1"'] & T^*Q \\ & M_1 \arrow[u, "j_1"'] \end{tikzcd}$$

commutes. For an almost regular Lagrangian one can prove that, in a pointwise sense,

(3.5) $$(FL_1)_*(\operatorname{char} \omega_L) = \operatorname{char} \Omega_1,$$

(cf. [5], p. 139).

Returning to the equation (3.3) it is to be noticed that the solvability conditions (3.4) in general will not be satisfied. The constraint algorithm then generates a sequence of sumbanifolds $\ldots \to M_l \to \ldots \to M_2 \to M_1$ which are defined by

$$M_l = \{m \in M_{l-1} \mid \langle \xi, \mathrm{d}H_1(m)\rangle = 0 \quad \text{for all} \quad \xi \in T_m M_{l-1}^{\S}\}$$

where

$$T_m M_{l-1}^{\S} = \{\xi \in T_m M_1 \mid (\Omega_1)_m(\xi, \eta) = 0 \quad \text{for all} \quad \eta \in T_m M_{l-1}\}.$$

The submanifolds $M_2, M_3, \ldots$ are called secondary, tertiary, ... constraint submanifolds. The sequence eventually terminates at some final constraint submanifold M. If $M \neq \phi$, then the equation $(Y \lrcorner \Omega_1 + \mathrm{d}H_1)|_M = 0$ possesses at least one consistent solution. For details, see [2] and [5].

In terms of this constraint algorithm it has been demonstrated in [5] that for almost regular Lagrangians the Lagrangian description of the system always admits an equivalent Hamiltonian description. A special instance of this equivalence result is given by the following proposition.

PROPOSITION 3.1. *An almost regular Lagrangian admits a global dynamics (in the sense of definition 2.2) if and only if there are only primary constraints.*

Proof. If there are only primary constraints, i.e. if the constraint algorithm terminates at M_1, then the equation (3.3) admits a solution. Any such solution can then be lifted by the submersion FL_1 to a vector field on TQ which will satisfy the dynamical equation $X \lrcorner\, \omega_L = -\,\mathrm{d}E_L$ (cf. [5]). Conversely, suppose L admits a global dynamics. Then for any $\hat{\xi} \in \operatorname{char} \omega_L$,

$$\langle \hat{\xi}, \mathrm{d}E_L(\tau_{Q*}(\hat{\xi}))\rangle = 0. \tag{3.6}$$

Take $m \in M_1$ and $x \in FL_1^{-1}\{m\}$. According to (3.5), for any $\xi \in T_m M_1^{\S}$ (= char $\Omega_1|_{\{m\}}$) there exists a $\hat{\xi} \in \operatorname{char} \omega_L$ with $\tau_{Q*}(\hat{\xi}) = x$ and such that

$$(FL_1)_*(\hat{\xi}) = \xi.$$

In view of (3.2) and (3.6) we then obtain

$$\begin{aligned} 0 = \langle \hat{\xi}, \mathrm{d}E_L(x)\rangle = \langle \hat{\xi}, [FL_1^*(\mathrm{d}H_1)](x)\rangle &= \\ = \langle (FL_1)_*\hat{\xi}, \mathrm{d}H_1(m)\rangle &= \\ = \langle \xi, \mathrm{d}H_1(m)\rangle&. \end{aligned}$$

Hence, the conditions (3.4) are satisfied and so there are no secondary constraints. ∎

Assuming the foliation defined by char Ω_1 on M_1 is a fibration we have a symplectic reduction of (M_1, Ω_1), i.e., the quotient space $M_1/\operatorname{char} \Omega_1$ admits a manifold structure and there exists a unique symplectic form $\widetilde{\Omega}$ on $M_1/\operatorname{char} \Omega_1$ for which

$$\pi_1^* \widetilde{\Omega} = \Omega_1,$$

with $\pi_1 : M_1 \to M_1/\operatorname{char} \Omega_1$ the natural projection (cf. [16], theorem 25.2). In case there are no secondary constraints, the function H_1 is π_1-projectable with projection $\widetilde{H}_1$ and any solution of (3.3) projects onto the Hamiltonian vector field $\widetilde{Y}$ which satisfies

$$\widetilde{Y} \lrcorner\, \widetilde{\Omega} = -\,\mathrm{d}\widetilde{H}_1.$$

The purpose of the subsequent analysis can now be formulated as follows.

Suppose we are given an almost regular Lagrangian L on TQ which satisfies the conditions of theorem 2.3, such that there exists a reduced regular Lagrangian description on $TQ/\operatorname{char} \omega_L$. What is then the relation between this reduced Lagrangian system and the reduced Hamiltonian system obtained via the constraint algorithm.

Before dealing with this problem we will first describe a general property of almost regular type II Lagrangians. (In the sequel, it will again always be tacitly assumed that the Lagrangians under consideration satisfy (A1), (A2) and (A3) from section 2.1).

Let us now fix some notation. The orthogonal (or symplectic) complement of T_mM_1, regarded as a subspace of $T_m(T^*Q)$, with respect to the canonical symplectic form Ω_Q will be denoted by $T_mM_1^{\perp}$. We have the relation

$$T_mM_1^{\S} = T_mM_1^{\perp} \cap T_mM_1, \tag{3.7}$$

where, as above, $T_mM_1^{\S}$ stands for the orthogonal complement of T_mM_1 with respect to Ω_1 $(= \Omega_Q|_{M_1})$. Moreover,

$$\dim (T_mM_1^{\perp}) + \dim (T_mM_1) = 2n \tag{3.8}$$

where $n = \dim Q$ (see e.g. [19]). By definition, M_1 is a coisotropic submanifold of T^*Q if $T_mM_1^{\perp} \subset T_mM_1$ for each $m \in M_1$ which, in view of (3.7), is equivalent to

$$T_mM_1^{\perp} = T_mM_1^{\S}.$$

We now have the following important result:

PROPOSITION 3.2. *If L is an almost regular type II Lagrangian then M_1 is a coisotropic submanifold of T^*Q and the Legendre map induces a local symplectomorphism between* $(TQ/\text{char}\ \omega_L, \tilde{\omega})$ *and* $(M_1/\text{char}\ \Omega_1, \tilde{\Omega})$.

Proof. The proof that M_1 is coisotropic is mainly a matter of counting dimensions. We already noticed before that for an almost regular Lagrangian the primary constraint submanifold M_1 is diffeomorphic to $TQ/V(\text{char}\ \omega_L)$, from which we deduce that $\dim [V(\text{char}\ \omega_L)] + \dim (M_1) = \dim (TQ) = 2n$. Combining this with (3.8) we find

$$\dim [V(\text{char}\ \omega_L)] = \dim (T_mM_1^{\perp}) \tag{3.9}$$

for any $m \in M_1$. Since FL is a submersion onto its image so is FL_1 and, obviously, $\text{Ker}\,(FL_*) = \text{Ker}\,(FL_{1*})$. Using (3.1) and (3.5) the following dimensional relation is then easily verified:

$$\dim (\text{char}\ \omega_L) = \dim (T_mM_1^{\S}) + \dim [V(\text{char}\ \omega_L)]. \tag{3.10}$$

By definition of a type II Lagrangian, $\dim (\text{char}\ \omega_L) = 2 \dim [V(\text{char}\ \omega_L)]$ and so it follows from (3.9) and (3.10) that $\dim (T_mM_1^{\S}) = \dim (T_mM_1^{\perp})$ which in view of (3.7) implies $T_mM_1^{\S} = T_mM_1^{\perp}$. Since this is true for any $m \in M_1$, M_1 is coisotropic. To prove the second assertion we observe that since FL_1 maps char ω_L onto char Ω_1 (cf. (3.5)), it induces a map $\widetilde{FL} : TQ/\text{char}\ \omega_L \to M_1/\text{char}\ \Omega_1$ such that

$$\pi_1 \circ FL_1 = \widetilde{FL} \circ \pi_L.$$

We then have

$$\begin{aligned}\pi_L^* (\widetilde{FL}^*\widetilde{\Omega}) &= FL_1^*(\pi_1^*\widetilde{\Omega}) = \\ &= FL_1^*(\Omega_1) = \\ &= \omega_L\end{aligned}$$

from which it follows that $\widetilde{FL}^*\widetilde{\Omega} = \tilde{\omega}$, and this completes the proof. ∎

Remarks

1. It is interesting to observe that the converse of the first assertion of this proposition is also true, namely: if for an almost regular Lagrangian L the primary constraint submanifold M_1 is coisotropic, then L is of type II. The proof follows immediately from (3.9) and (3.10).

2. The situation described in the previous proposition can be summarized by the following commutative diagram

$$\begin{tikzcd}
(TQ, \omega_L) \arrow[rr, "FL"] \arrow[drr, "FL_1"] \arrow[dd, "\pi_L"'] & & (T^*Q, \Omega_Q) \\
& & (M_1, \Omega_1) \arrow[u, "j_1"'] \arrow[d, "\pi_1"] \\
(TQ/\mathrm{char}\,\omega_L, \tilde{\omega}) \arrow[rr, "\widetilde{FL}"'] & & (M_1/\mathrm{char}\,\Omega_1, \widetilde{\Omega}).
\end{tikzcd}$$

In the next section we will specialize this picture to the case where the given system admits a regular Lagrangian reduction, thereby elucidating the nature of the induced map $\widetilde{FL}$.

3.2. Invariant Lagrangians for which char ω_L is a tangent distribution

Let L be a degenerate Lagrangian for which char ω_L is the tangent distribution of an integrable distribution $\mathcal{D}$ on Q and assume that the foliation defined by $\mathcal{D}$ is a fibration. From the analysis in section 2.5 we then know that on the leaf space of char ω_L, which can here be identified with $T(Q/\mathcal{D})$, there exists a reduced regular Lagrangian system. Moreover, if L is π_L-projectable, i.e., if L is invariant under char ω_L, then its projection $\tilde{L}$ defines a regular Lagrangian for the reduced system.

According to proposition 2.8, L will be invariant under char ω_L if and only if θ_L is projectable and the latter in turn is equivalent with $X^v(L) = 0$ for each $X \in \mathcal{D}$ (cf. (2.14)). It will be seen that this invariance condition has important consequences for the structure of the Legendre map FL.

Before proceeding we first recall that to each vector field X on Q one can associate a Hamiltonian vector field on T^*Q which projects onto X and leaves the canonical 1-form θ_Q invariant. This lifted vector field, which is uniquely defined, is called the complete or Hamiltonian lift of X to T^*Q and will be denoted by X^{c*}. In coordinates, if $X = \lambda^i\, \partial/\partial q^i$ then, with respect to the canonical coordinates on T^*Q the expression for X^{c*} is

$$X^{c*} = \lambda^i \frac{\partial}{\partial q^i} - p_j \frac{\partial \lambda^j}{\partial q^i} \frac{\partial}{\partial p_i}.$$

LEMMA 3.1. *Let L be a degenerate Lagrangian for which char ω_L is the tangent distribution of a distribution $\mathcal{D}$ on Q and suppose $X^v(L) = 0$ for each $X \in \mathcal{D}$. Then, the complete and the Hamiltonian lifts X^c and X^{c*} of a vector field X in $\mathcal{D}$ are FL-related, i.e.*

$$FL_* \circ X^c = X^{c*} \circ FL.$$

Proof. The proof can be most easily established by means of a simple coordinate calculation. Representing X by $\lambda^i\, \partial/\partial q^i$ we have $X^c = \lambda^i\, \partial/\partial q^i + v^j(\partial\lambda^i/\partial q^j)(\partial/\partial v^i)$ and $X^{c*} = \lambda^i\, \partial/\partial q^i - p_j(\partial\lambda^j/\partial q^i)(\partial/\partial p_i)$.

If X belongs to $\mathcal{D}$ then $X^c \in$ char ω_L, i.e. $X^c \lrcorner\, \omega_L = 0$. Using the local expression (2.3b) for ω_L this leads to

$$\lambda^j \frac{\partial^2 L}{\partial v^i\, \partial v^j} = 0$$

and

$$\lambda^j \left(\frac{\partial^2 L}{\partial v^i\, \partial q^j} - \frac{\partial^2 L}{\partial v^j\, \partial q^i} \right) + v^j \frac{\partial \lambda^k}{\partial q^j} \frac{\partial^2 L}{\partial v^k\, \partial v^i} = 0 \tag{3.11}$$

for $i = 1, 2, \ldots, n$ ($= \dim Q$). For a fixed point (q, v) of TQ we compute the image of the vector $X^c(q, v)$ under FL_*. With a slight abuse of notation we obtain

$$FL_*(X^c(q,v)) = \lambda^i(q) \frac{\partial}{\partial q^i} + \left(\lambda^j \frac{\partial^2 L}{\partial q^j\, \partial v^i} + v^j \frac{\partial \lambda^k}{\partial q^j} \frac{\partial^2 L}{\partial v^k\, \partial v^i} \right)(q,v) \frac{\partial}{\partial p_i}$$

which in view of (3.11) reduces to

$$FL_*(X^c(q,v)) = \lambda^i(q) \frac{\partial}{\partial q^i} + \lambda^j(q) \frac{\partial^2 L}{\partial v^j\, \partial q^i}(q,v) \frac{\partial}{\partial p_i}. \tag{3.12}$$

The right-hand side here represents a tangent vector to T^*Q at the point

$$FL(q, v) = \left(q, \frac{\partial L}{\partial v}(q, v)\right).$$

By assumption $X^v(L) = 0$, i.e. $\lambda^i\, \partial L/\partial v^i = 0$. Taking this into account, (3.12) becomes

$$FL_*(X^c(q,v)) = \lambda^i(q)\frac{\partial}{\partial q^i} - \frac{\partial L}{\partial v^j}(q, v)\frac{\partial \lambda^j}{\partial q^i}(q)\frac{\partial}{\partial p_i}$$

which precisely equals $X^{c*}(FL(q, v))$, as required. ∎

The annihilator of the distribution $\mathcal{D}$ is the subbundle of T^*Q defined by

$$\mathrm{Ann}\,\mathcal{D} = \{\alpha \in T^*Q \mid \langle X(\pi_Q(\alpha)), \alpha\rangle = 0 \quad \text{for all} \quad X \in \mathcal{D}\}$$

where $\pi_Q : T^*Q \to Q$ denotes the cotangent bundle projection. The codimension of Ann $\mathcal{D}$ is equal to the dimension of $\mathcal{D}$.

LEMMA 3.2. *Let L be a degenerate Lagrangian for which* char ω_L *is the tangent distribution of $\mathcal{D}$, then* Im $(FL) \subset$ Ann $\mathcal{D}$ *if and only if $X^v(L) = 0$ for each $X \in \mathcal{D}$.*

Proof. By definition of Ann $\mathcal{D}$ the condition Im $(FL) \subset$ Ann $\mathcal{D}$ means that for each point (q, v) in TQ and each $X \in \mathcal{D}$,

$$\langle X(q), FL(q, v)\rangle = 0.$$

In coordinates, with $X = \lambda^i \dfrac{\partial}{\partial q^i}$, this still reads

$$\lambda^i(q)\frac{\partial L}{\partial v^i}(q, v) = 0,$$

which is precisely the coordinate expression of $X^v(L) = 0$. ∎

The two previous lemmas actually hold without having to impose the almost regualrity condition on L. We now bring this condition into the picture again. Char ω_L being a tangent distribution, L is of type II and we therefore know from proposition 3.2 that Im $(FL) = M_1$ is a coisotropic submanifold of T^*Q. Moreover, we have

PROPOSITION 3.3. *Under the assumptions of lemma 3.1, with L almost regular, M_1 is an open submanifold of* Ann $\mathcal{D}$ *and the characteristic distribution of Ω_1 ($= \Omega_Q|_{M_1}$) is spanned by the Hamiltonian lifts of the vector fields in $\mathcal{D}$.*

Proof. From lemma 3.2 we already know that $M_1 \subset \mathrm{Ann}\, \mathcal{D}$. For M_1 to be an open submanifold of $\mathrm{Ann}\, \mathcal{D}$ it therefore remains to be verified that both spaces have the same dimension. Since $\mathrm{codim}\,(M_1) = \dim\,(\mathrm{Ker}\,(FL_*))$ and taking into account that $\mathrm{Ker}\,(FL_*) = V(\mathrm{char}\, \omega_L)$, it readily follows that $\dim\,(\mathcal{D}) = \mathrm{codim}\,(M_1)$. On the other hand, $\dim\,(\mathcal{D}) = \mathrm{codim}\,(\mathrm{Ann}\, \mathcal{D})$ whence $\dim\,(\mathrm{Ann}\, \mathcal{D}) = \dim\,(M_1)$.

To prove the second assertion we first observe that $FL_*\,(\mathrm{char}\, \omega_L)$ defines a distribution on M_1 which, according to (3.5), is precisely the characteristic distribution of Ω_1. Again using the fact that $\mathrm{Ker}\,(FL_*) = V(\mathrm{char}\, \omega_L)$ one then easily deduces from lemma 3.1 that $FL_*(\mathrm{char}\, \omega_L)$ is spanned by the restrictions to M_1 of the lifts X^{c*} of the vector fields $X \in \mathcal{D}$. ∎

The Hamiltonian lifts of the vector fields in $\mathcal{D}$ define a distribution on T^*Q which we will denote by $\mathcal{D}^{c*}$. Integrability of $\mathcal{D}^{c*}$ follows from integrability of $\mathcal{D}$ in view of the property $[X^{c*}, Y^{c*}] = [X, Y]^{c*}$ which holds for any two vector fields X, Y on Q.

Next, we notice that one can associative with the given distribution $\mathcal{D}$ a «momentum map» J which, at least in a formal way, can be defined as the rule which assigns to each point $\alpha \in T^*Q$ a $\mathbb{R}$-linear function $J(\alpha)$ on $\mathcal{D}$ such that

$$J(\alpha)(X) = \langle X(\pi_Q(\alpha)), \alpha \rangle, \qquad \text{for } X \in \mathcal{D}.$$

For each $X \in \mathcal{D}$, the corresponding function $J_X : T^*Q \to \mathbb{R}$, $\alpha \to J_X(\alpha) = J(\alpha)(X)$ is the Hamiltonian of the lifted vector field X^{c*}. J_X is also called the momentum corresponding to X (cf. [19], section 4.2).

Clearly, the zero-level set of this «momentum map» is precisely the annihilator of the distribution $\mathcal{D}$, i.e. $\mathrm{Ann}\, \mathcal{D} = J^{-1}\{0\}$. Proposition 3.3 now tells us that, under the given assumptions, FL maps TQ onto an open submanifold M_1 of $J^{-1}\{0\}$ and, moreover, $\mathcal{D}^{c*}$ restricts to an integrable distribution on M_1 namely, $\mathrm{char}\, \Omega_1$. One can then prove that the reduced phase space $(M_1/\mathrm{char}\, \Omega_1, \widetilde{\Omega})$ is symplectomorphic to an open submanifold of $(T^*(Q/\mathcal{D}), \Omega_{Q/\mathcal{D}})$ where $\Omega_{Q/\mathcal{D}}$ is the canonical symplectic form on $T^*(Q/\mathcal{D})$. (For a proof one can proceed along the same lines as in [19], theorem 4.3.3, which deals with Hamiltonian actions of a Lie group on a cotangent bundle).

On the other hand we know that, in the case under consideration, $TQ/\mathrm{char}\, \omega_L \cong T(Q/\mathcal{D})$, and taking into account the assumed invariance of L under $\mathrm{char}\, \omega_L$, it follows that the induced symplectic form $\widetilde{\omega}$ can be identified with the Poincaré-Cartan 2-form $\omega_{\widetilde{L}} = d\theta_{\widetilde{L}}$ of the regular projected Lagrangian $\widetilde{L}$.

Returing to proposition 3.2 and bearing in mind the above identifications, it can now easily be shown that the local symplectomorphism $\widetilde{FL}$ coincides with the Legendre map $F\widetilde{L}$ corresponding to $\widetilde{L}$. To be more precise, we actually have $F\tilde{L} = \tilde{j} \circ \widetilde{FL}$, with $\tilde{j}$ the symplectic embedding of $M_1/\mathrm{char}\,\Omega_1$ into $T^*(Q/\mathcal{D})$. We can summarize the situation in the following commutative diagram:

$$\begin{tikzcd}
(TQ, \omega_L) \arrow[rr, "FL"] \arrow[ddd, "\pi_L"'] \arrow[drr, "FL_1"] & & (T^*Q, \Omega_Q) \\
& & (M_1, \Omega_1) \arrow[u, "j_1"'] \arrow[d, "\pi_1"] \\
& & (M_1/\mathrm{char}\,\Omega_1, \tilde{\Omega}) \arrow[d, "\tilde{j}"] \\
(TQ/\mathrm{char}\,\omega_L, \tilde{\omega}) \cong (T(Q/\mathcal{D}), \omega_{\tilde{L}}) \arrow[rr, "F\tilde{L}"'] \arrow[urr, "\widetilde{FL}"] & & (T^*(Q/\mathcal{D}), \Omega_{Q/\mathcal{D}})
\end{tikzcd}$$

In the previous subsection we have seen that the constraint algorithm produces a Hamiltonian system $\tilde{Y}$ on the reduced phase space $M_1/\mathrm{char}\,\Omega_1$ with Hamiltonian $\tilde{H}_1$, where $\tilde{H}_1 \circ \pi_1 = H_1$ and $H_1 \circ FL = E_L$. Through the symplectic embedding $\tilde{j}$ this can still be identified with a Hamiltonian system on (an open submanifold of) $T^*(Q/\mathcal{D})$ for which, for simplicity, we retain the same notation. By means of the above commutative scheme one then easily deduces that $\tilde{H}_1 \circ F\tilde{L} = E_{\tilde{L}}$. Since, moreover, $F\tilde{L}^*\Omega_{Q/\mathcal{D}} = \omega_{\tilde{L}}$, it follows that the reduced Lagrangian system $\tilde{\Gamma}$ and the reduced Hamiltonian system $\tilde{Y}$, associated with the given degenerate Lagrangian L, are $F\tilde{L}$-related, i.e.

$$F\tilde{L}_* \circ \tilde{\Gamma} = \tilde{Y} \circ F\tilde{L}.$$

Herewith we have proven the following result which completes the reduction picture in the case of a degenerate Lagrangian which is invariant under char ω_L, with char ω_L a tangent distribution.

THEOREM 3.1. *Let L be an almost regular Lagrangian for which* char ω_L *is the tangent distribution of a distribution $\mathcal{D}$ and suppose $X^v(L) = 0$ for each $X \in \mathcal{D}$. Then, the reduced regular Lagrangian system is locally symplectomorphic to the Hamiltonian system which, according to Dirac's theory of constraints, is induced on the reduced phase space. Such a local symplectomorphism is provided by the Legendre map $F\tilde{L} : T(Q/\mathcal{D}) \to T^*(Q/\mathcal{D})$ corresponding to the regular projected Lagrangian $\tilde{L}$.*

3.3. The reduction of Hamiltonian systems with symmetry

The reduction procedure of Marsden and Weinstein provides a very nice description of the reduction of dynamical systems with symmetry on a symplectic manifold ([3], [18]). Let us briefly recall how the method works.

Let G be a connected Lie group which acts smoothly on a connected manifold Q. The lifted action of G on T^*Q is a Hamiltonian action with an Ad*-equivariant momentum map $J : T^*Q \to g^*$, where g^* is the dual of the Lie algebra g of G. For a given element α of T^*Q, $J(\alpha)$ is defined by

$$J(\alpha)(a) = \langle (X_a)^{c*}, \theta_Q \rangle (\alpha) =$$

$$= \langle X_a(\pi_Q(\alpha)), \alpha \rangle,$$

for each $a \in g$, where X_a is the infinitesimal generator of the action of G on Q corresponding to a.

For $\mu \in g^*$ we denote by G_μ the isotropy group of μ under the co-adjoint action Ad* of G on g^*. For each regular value μ of J the subgroup G_μ of G acts on the submanifold $J^{-1}\{\mu\}$ of T^*Q and one can construct the orbit space $F_\mu = J^{-1}\{\mu\}/G_\mu$ with projection $\pi_\mu : J^{-1}\{\mu\} \to F_\mu$. Assuming the action of G_μ on $J^{-1}\{\mu\}$ is free and proper, F_μ can be equipped with a manifold structure and there exists a unique symplectic form $\tilde{\Omega}_\mu$ on F_μ such that $\pi_\mu^* \tilde{\Omega}_\mu = j_\mu^* \Omega_Q$, where $j_\mu : J^{-1}\{\mu\} \to T^*Q$ is the inclusion. The symplectic manifold $(F_\mu, \tilde{\Omega}_\mu)$ is called the reduced phase space corresponding to μ.

Consider now a Hamiltonian system on T^*Q with Hamiltonian H and suppose H is invariant under the action of G on T^*Q. Then there exists a reduced Hamiltonian system on F_μ with Hamiltonian H_μ such that $\pi_\mu^* H_\mu = j_\mu^* H$. (For more details, see also for instance [16], [19] and [24]).

If in the preceding we take $\mu = 0$ then $G_\mu = G$ and it follows from the theory of the momentum map that the zero-level set $J^{-1}\{0\} = \mathrm{Ann}\, g$, where Ann g is a shorthand notation for the annihilator of the distribution defined by the orbits of G on Q. The resulting picture is now strongly reminiscent of the one we have met in the previous subsection when dealing with the reduction of a special class of degenerate Lagrangians.

In fact, we will show that theorem 3.1 establishes a link between the reduction scheme developed above for degenerate Lagrangians on the one hand, and the Marsden-Weinstein reduction (with respect to the zero regular value of the momentum map) on the other hand. Alternatively, it can also be said that theorem 3.1 provides us with an interpretation of Hamiltonian systems with symmetry in terms of degenerate Lagrangians. This will in particular be illustrated in the case of systems of mechanical type (i.e. systems with a Hamiltonian of the form kinetic plus potential energy).

For simplicity, the Lie algebra of the infinitesimal generators of the action of G on the base manifold Q will also be denoted by $\mathfrak{g}$. For a free action, this algebra is in fact (anti-)isomorphic to the Lie algebra of G. The tangent distribution of $\mathfrak{g}$ is then the distribution on TQ spanned by its tangent algebra $T\mathfrak{g}$, i.e., the algebra defined by the complete and vertical lifts of elements of $\mathfrak{g}$ (cf. section 2.2).

The following proposition already indicates the link between the Marsden--Weinstein reduction procedure and the reduction induced by a degenerate Lagrangian.

PROPOSITION 3.4. *Let G be a Lie group which acts freely and properly on a connected manifold Q. Then there exists a degenerate Lagrangian L on TQ such that* char ω_L *is precisely the tangent distribution of $\mathfrak{g}$. Moreover, the reduced phase space corresponding to the zero regular value of the momentum map of the lifted action of G on T^*Q is symplectomorphic to the leaf space $TQ/$*char ω_L.

Proof. Since G acts freely and properly on Q, the leaf space Q/G admits a manifold structure. Let $\tilde{g}$ be a nondegenerate metric on Q/G and define g as the pull-back of $\tilde{g}$ under the projection of Q onto Q/G. Then g is a degenerate metric on Q and we denote its null distribution by N (i.e. N is spanned by the vector fields X on Q satisfying $g(X, .) = 0$).

Let L be the kinetic energy Lagrangian corresponding to g, i.e., in coordinates: $L(q, v) = \frac{1}{2}\, g_q(v, v)$. This type of Lagrangian, defined in terms of a degenerate metric, will be discussed in detail in the next section.

By construction of g its null distribution is spanned by $\mathfrak{g}$ and consists of Killing vector fields. From the analysis in section 4.2 it then follows that N is integrable and char ω_L is its tangent distribution.

The remainder of the proof is now a straightforward application of theorem 3.1. In particular, we notice that in the present case $\mathrm{Im}\,(FL) = J^{-1}\{0\}$, with J the momentum map of the action of G on T^*Q. The reduced regular Lagrangian $\tilde{L}$ which is here the kinetic energy corresponding to the given nondegenerate metric $\tilde{g}$ on Q/G, is hyperregular and therefore induces a symplectomorphism between $(T(Q/G), \omega_{\tilde{L}})$ and $(T^*(Q/G), \Omega_{Q/G})$. Finally, it can also be shown that the reduced phase space $(J^{-1}\{0\}/G, \tilde{\Omega}_0)$ is symplectomorphic to $(T^*(Q/G), \Omega_{Q/G})$, (this follows for instance from [19], theorem 4.3.3). ∎

Before continuing we first mention the following interesting property:

LEMMA 3.3. *Let L be a Lagrangian which is invariant under the tangent distribution of a distribution $\mathcal{D}$ on Q, i.e. $X^v(L) = X^c(L) = 0$ for each $X \in \mathcal{D}$. Then, L is degenerate and* char ω_L *contains the tangent distribution of $\mathcal{D}$.*

Proof. Since $\mathcal{L}_{X^v} S = \mathcal{L}_{X^c} S = 0$ and taking into account the assumed invariance of L it follows that for each $X \in \mathcal{D}$

$$\mathcal{L}_{X^c} \theta_L = \mathcal{L}_{X^c} (\mathrm{d}L \circ S) = 0$$

and

$$\mathcal{L}_{X^v} \theta_L = \mathcal{L}_{X^v} (\mathrm{d}L \circ S) = 0.$$

Hence,

$$0 = X^c \,\lrcorner\, \omega_L + \mathrm{d}(X^c \,\lrcorner\, \theta_L) = X^c \,\lrcorner\, \omega_L + \mathrm{d}X^v(L) = X^c \,\lrcorner\, \omega_L$$

and

$$0 = X^v \,\lrcorner\, \omega_L + \mathrm{d}(X^v \,\lrcorner\, \theta_L) = X^v \,\lrcorner\, \omega_L,$$

which proves that char ω_L contains the complete and vertical lifts of vector fields in $\mathcal{D}$. ∎

We now turn our attention to simple mechanical systems with symmetry. A detailed analysis of these systems can be found for instance in [19], section 4.5. Recall that a simple mechanical system with symmetry may be characterized by a quadruple (Q, g, V, G) where: Q is a (connected) Riemannian manifold with Riemannian metric g, which determines the kinetic energy of the system; V is a smooth function on Q representing the potential energy of the system and G is a connected Lie group acting on Q such that both g and V are invariant under this action. The Hamiltonian of the system then reads

$$H = K + V \circ \pi_Q$$

where K is (the phase space expression of) the kinetic energy, i.e., for $\alpha \in T_m^*Q$: $: K(\alpha) = g_m(\alpha^\#, \alpha^\#)$ with $\alpha^\# \in T_m Q$ the inverse image of α under the isomorphism $g_m : T_m Q \to T_m^* Q$, $v \to g_m(v, .)$. Under the given assumptions H is invariant under the lifted action of G on T^*Q and thus one can apply the Marsden-Weinstein procedure for constructing a reduced Hamiltonian system.

In particular, we are interested here in the reduced system corresponding to the zero regular value of the momentum map J of the action of G on T^*Q. As before, the action of G on Q is assumed to be free and proper.

We then have:

THEOREM 3.2. *There exists a degenerate Lagrangian L' on TQ such that: (i)* char $\omega_{L'}$ *is the tangent distribution of* $\mathcal{G}$*; (ii)* L' *admits a regular Lagrangian reduction which is symplectomorphic to the reduced Hamiltonian system corresponding to the zero regular value of the momentum map.*

Proof. Under the given assumptions, the Riemannian metric g and the potential energy function V both pass to the quotient Q/G, with projections denoted by $\tilde{g}$ and $\tilde{V}$ respectively. In particular, $\tilde{g}$ is a Riemannian metric on Q/G such that the projection $\pi : Q \to Q/G$ becomes a Riemannian submersion.

On $T(Q/G)$ we consider the Lagrangian $\tilde{L} = \tilde{K} - \tilde{V} \circ \tau_{Q/G}$, where $\tilde{K}$ stands for the kinetic energy function associated with $\tilde{g}$. $\tilde{L}$ is hyperregular and $\omega_{\tilde{L}}$ is symplectic. We now define the Lagrangian L' on TQ by

$$L' = \tilde{L} \circ \pi_{*}.$$

Clearly, L' is invariant under the tangent distribution of $\mathcal{G}$, i.e., $X^{v}(L') = X^{c}(L') = 0$ for each infinitesimal generator X of the action G on Q. Taking into account lemma 3.3 and the fact that $\omega_{L'}$ projects onto the symplectic form $\omega_{\tilde{L}}$, this proves (i).

In order to prove (ii) we first observe that $\operatorname{Im} FL' = J^{-1}\{0\}$. (Since the potential energy term in L' has no effect on the Legendre map FL', the situation is similar to the one described in proposition 3.4). Moreover, it is easy to verify that $H \circ FL' = E_{L'}$, with H the Hamiltonian of the given mechanical system. The remainder of the proof then follows from theorem 3.1. ∎

4. EXAMPLES AND APPLICATIONS

In this section we will study a few typical examples of degenerate Lagrangians, using the formalism developed in the previous sections, and discuss some related aspects. Special attention will be paid to Lagrangians of 'kinetic energy type', defined in terms of a degenerate metric, and to the homogeneous description of time-dependent Lagrangian systems.

Throughout this section we will mainly use coordinate expressions.

4.1. Lagrangians defined by a degenerate metric

Let g be a symmetric type (0,2) tensor field on Q. For each $m \in Q$ we define a subspace N_m of $T_m Q$ by

$$N_m = \{x \in T_m Q \,|\, g_m(x, y) = 0 \quad \text{for all} \quad y \in T_m Q\}.$$

Suppose that $\dim N_m$ is not zero and is the same for all $m \in Q$, and that the bilinear form induced by g_m on $T_m Q / N_m$ is positive definite: then we call g a degenerate metric on Q and $N = \bigcup_{m \in Q} N_m$ its null distribution.

It is known (see [25]) that the necessary and sufficient condition for a degenerate metric to admit a compatible symmetric connection is that $\mathcal{L}_Z g = 0$ for all vector fields $Z \in N$ (i.e. for all smooth sections of N). Under this condition N is integrable in the sense of Frobenius' theorem since for any $Y, Z \in N$ and any vector field X on Q

$$g([Y, Z], X) = Y(g(Z, X)) - g(Z, [Y, X]) = 0.$$

However, this integrability may hold even though vector fields in N do not satisfy the «isometry» or «Killing» condition $\mathcal{L}_Z g = 0$. For example, if $g = G(q^1) dq^2 \otimes \otimes dq^2$ on $\mathbb{R}^2$ with G a positive function, then N is the 1-dimensional distribution spanned by $\partial/\partial q^1$ and thus is certainly integrable, but $\mathcal{L}_{\partial/\partial q^1} g \neq 0$ unless G is a constant.

The condition that N be integrable is, despite appearances, a purely tensorial one. Indeed, if $Y, Z \in N$ then for any vector field X and any function f on Q

$$g([Y, fZ], X) = g(f[Y, Z], X) + g(Y(f)Z, X) =$$
$$= fg([Y, Z], X).$$

This last observation is also easily verified in coordinates. Suppose $Y = \eta^i\, \partial/\partial q^i$ and $Z = \zeta^i\, \partial/\partial q^i$, then for $Y, Z \in N$

$$g_{ij}\eta^i = g_{ij}\zeta^i = 0, \tag{4.1}$$

where g_{ij} are the components of g. The condition for integrability of N is that for each $Y, Z \in N$

$$g_{ij}\left(\eta^k \frac{\partial \zeta^j}{\partial q^k} - \zeta^k \frac{\partial \eta^j}{\partial q^k}\right) = 0,$$

which, using (4.1), may be rewritten as

$$\eta^j \zeta^k \left(\frac{\partial g_{ij}}{\partial q^k} - \frac{\partial g_{ik}}{\partial q^j}\right) = 0.$$

The condition for $Z = \zeta^i\, \partial/\partial q^i$ to be a Killing vector field, i.e. $\mathcal{L}_Z g = 0$, in coordinates is

$$\zeta^k \left(\frac{\partial g_{ij}}{\partial q^k} - \frac{\partial g_{ik}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^i}\right) = 0.$$

Consider the Lagrangian L defined in terms of a degenerate metric g on Q by

$$L(q, v) = \frac{1}{2} g_q(v, v) = \frac{1}{2} g_{ij}(q)\, v^i v^j.$$

We shall first examine the conditions under which L is of type II. The expressions (2.3a) and (2.3b) for θ_L and ω_L here become

$$\theta_L = g_{ij}\, v^i\, \mathrm{d}q^j, \qquad \omega_L = g_{ij}\, \mathrm{d}v^i \wedge \mathrm{d}q^j + \frac{\partial g_{jk}}{\partial q^i}\, v^k \mathrm{d}q^i \wedge \mathrm{d}q^j.$$

A vector field $Z = \zeta^i\, \partial/\partial q^i + \nu^i\, \partial/\partial v^i$ belongs to char ω_L iff

(4.2a) $$g_{ij}\, \zeta^j = 0,$$

(4.2b) $$g_{ij}\, \nu^j = \zeta^j \left(\frac{\partial g_{ij}}{\partial q^k} - \frac{\partial g_{ik}}{\partial q^j} \right) v^k.$$

From this one can already deduce that, in any event, $V(\text{char}\, \omega_L)$ is spanned by the vertical lifts of vector fields in N. In view of lemma 2.1 and taking account of (2.5) it follows that L will be of type II if and only if $V(\text{char}\, \omega_L) \subset S(\text{char}\, \omega_L)$, which here means that (4.2b) must admit a solution for ν^j for every choice of ζ^j satisfying (4.2a). Now, the necessary and sufficient condition for (4.2b) to admit a solution is that the right-hand side gives zero when contracted with any η^i such that $g_{ij}\eta^i = 0$. Since this must hold for every v the condition is that

$$\eta^j \zeta^k \left(\frac{\partial g_{ij}}{\partial q^k} - \frac{\partial g_{ik}}{\partial q^j} \right) = 0$$

whenever $g_{ij}\eta^j = g_{ik}\zeta^k = 0$ and this is equivalent to demanding that N be integrable (see above). Summarizing, we have thus proved the following property:

PROPOSITION 4.1. *The kinetic energy Lagrangian corresponding to a degenerate metric g is of type II if and only if the null distribution of g is integrable.* ■

Assuming that L is of type II we now ask what further condition is required to ensure the existence of a global dynamics, that is, a vector field X on TQ satisfying $X \lrcorner\, \omega_L = -\,\mathrm{d}E_L$. In the present case we have $E_L = L = \frac{1}{2} g_{ij}\, v^i v^j$ and thus

$$\mathrm{d}E_L = g_{ij}\, v^j\, \mathrm{d}v^i + \frac{1}{2} \frac{\partial g_{jk}}{\partial q^i}\, v^j v^k\, \mathrm{d}q^i.$$

If $X = \lambda^i\, \partial/\partial q^i + \mu^i\, \partial/\partial v^i$ we then require that

$$g_{jk}\lambda^k = g_{jk}v^k$$

$$g_{jk}\mu^k = \left(\frac{\partial g_{ik}}{\partial q^j} - \frac{\partial g_{ij}}{\partial q^k}\right) v^i\lambda^k - \frac{1}{2}\frac{\partial g_{ik}}{\partial q^j} v^i v^k .$$

From the first of these equations we obtain $\lambda^k = v^k + \eta^k$ where $g_{jk}\eta^k = 0$. Then the second condition becomes

$$g_{jk}\mu^k = \left(\frac{1}{2}\frac{\partial g_{ik}}{\partial q^j} - \frac{\partial g_{ij}}{\partial q^k}\right) v^i v^k + \left(\frac{\partial g_{ik}}{\partial q^j} - \frac{\partial g_{ij}}{\partial q^k}\right) v^i \eta^k .$$

As before, the necessary and sufficient condition for this equation to admit a solution for μ^k is that the contraction of the right-hand side with any ζ^j for which $g_{ij}\zeta^j = 0$, gives zero. Bearing in mind the assumption of the integrability of N, for this to hold for all v we must have

$$\zeta^j\left(\frac{\partial g_{ik}}{\partial q^j} - \frac{\partial g_{ij}}{\partial q^k} - \frac{\partial g_{jk}}{\partial q^i}\right) = 0$$

for every ζ^j satisfying $g_{ij}\,\zeta^j = 0$. This is precisely the condition that $\mathcal{L}_Z g = 0$ for all $Z \in N$. Thus, if L is of type II in order for there to be a global dynamics it must be the case that all vector fields in N are Killing vector fields. Conversely, if that condition holds, L is necessarily of type II since N is integrable, and there is also a global dynamics. Incidentally, it follows from the above calculation that the solution X of the dynamical equation may then be chosen to be a second-order equation field, as theory predicts (cf. theorem 2.1).

If $\mathcal{L}_Z g = 0$ for all $Z \in N$, then using the 'Killing' condition, equation (4.2b) may be rewritten as

$$g_{ij}\,\nu^j = g_{ij}\,\frac{\partial \zeta^j}{\partial q^k}\,v^k$$

from which $\nu^j = v^k\ \partial\zeta^j/\partial q^k + \eta^j$, where $g_{ij}\eta^j = 0$. Consequently, in this case char ω_L is spanned by the complete and vertical lifts of vector fields in N, i.e. char ω_L is the tangent distribution of N. It then follows from the theory that we may carry out the reduction procedure and, provided the leaf space Q/N has a manifold structure, the reduced Lagrangian will simply be the regular Lagrangian on $T(Q/N)$ defined by the nondegenerate metric $\tilde{g}$ on Q/N, obtained by passing g to the quotient.

We pause here for a moment to consider in general the effect on θ_L of changing the Lagrangian L by a gauge term.

Let α be a closed 1-form on Q and let $\hat{\alpha}$ be the corresponding function induced on TQ. If $L' = L + \hat{\alpha}$, then

$$\theta_{L'} = \theta_L + \tau_Q^* \alpha,$$

and, since $d\alpha = 0$, $\omega_{L'} = \omega_L$. For any vector field $Z \in \operatorname{char} \omega_L = \operatorname{char} \omega_{L'}$ we then have

$$\langle Z, \theta_{L'} \rangle = \langle Z, \theta_L \rangle + \langle Z, \tau_Q^* \alpha \rangle.$$

So, even if L is a Lagrangian which does itself pass to the quotient, the gauge equivalent Lagrangian L' need not necessarily satisfy the condition $\langle Z, \theta_{L'} \rangle = 0$ for all $Z \in \operatorname{char} \omega_{L'}$, necessary for $\theta_{L'}$ to pass to the quotient. As an explicit example: let $L = \frac{1}{2} g(v, v)$, where g is a degenerate metric for which N consists of Killing vector fields. Let α be a closed 1-form on Q which does not vanish on N; then $L' = \frac{1}{2} g(v, v) + \hat{\alpha}$ is an illustration of the situation described above.

4.2. Application to the spinor regularization of the Kepler problem

An example of reduction associated with a degenerate metric arises in the context of the spinor regularization of the Kepler problem (see [26], [27], [28], [29] for a general treatment of this problem).

Consider the complex space $\mathbb{C}^2 - \{0\}$ with coordinates $z = (z^i)$, $i = 1, 2$, where z is to be regarded as a column vector. We will write $\overline{z}^i$ for the complex conjugate of z^i and the real and imaginary parts of z^i are denoted by $\mathcal{R}e\, z^i$ and $\mathcal{I}m\, z^i$, respectively. A metric on $\mathbb{C}^2 - \{0\}$ is given by

$$g_z(u, v) = |z|^2\, \mathcal{R}e\, \langle u, v \rangle - \mathcal{I}m\, \langle z, u \rangle\, \mathcal{I}m\, \langle z, v \rangle.$$

where u stands for the real tangent vector $u^i\, \partial/\partial z^i + \overline{u}^i\, \partial/\partial \overline{z}^i$ and should again be regarded as a column vector (and v likewise); $\langle\ ,\ \rangle$ stands for the hermitian inner product, i.e., $\langle u, v \rangle = u^\dagger v = \overline{u}^1 v^1 + \overline{u}^2 v^2$ ($\dagger$ indicating complex conjugate transpose) and $|z|^2 = \langle z, z \rangle$.

This metric is degenerate and has a one (real) dimensional nullspace N_z, spanned by iz, at each point z. The vector field iz is the infinitesimal generator of the action $z \to e^{it} z$ of $U(1)$ on $\mathbb{C}^2 - \{0\}$ and g is clearly invariant under this action. The Lagrangian $L(z, v) = \frac{1}{2} g_z(v, v)$ on $T(\mathbb{C}^2 - \{0\})$ is degenerate, as is any Lagrangian obtained by adding to it for example a potential function of the form $V(|z|)$.

We are here precisely in a situation as the one described in the previous subsection and, in particular, the Lagrangian system is reducible under the action of $U(1)$.

In fact, the Lagrangian L is simply obtained from the usual kinetic energy Lagrangian on $T(E^3 - \{0\})$ (where E^3 is the 3-dimensional Euclidean space with the standard metric) by means of the spinor correspondance.

Let

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

be the Pauli matrices. Consider the map

$$\Sigma : \mathbb{C}^2 - \{0\} \to E^3 - \{0\}, \qquad z \to \frac{1}{2} z^\dagger \underline{\sigma} z$$

where $\underline{\sigma}$ is the «3-vector» $(\sigma_1, \sigma_2, \sigma_3)$. The elements of $\mathbb{C}^2 - \{0\}$ are called «spinors» and Σ is the spinor correspondance. It represents $\mathbb{C}^2 - \{0\}$ as a principle fibre bundle over $E^3 - \{0\}$ with group and fiber $U(1)$, and the Lagrangian L reduces to the standard kinetic energy Lagrangian on $T(E^3 - \{0\})$.

The cotangent bundle $T^*(\mathbb{C}^2 - \{0\})$ has fibre coordinates $\rho = (\rho_1, \rho_2)$ where the ρ_i are to be regarded as components of a row vector and represent the real covector $\rho_i \, dz^i + \bar{\rho}_i \, d\bar{z}^i$. The canonical 1-form θ_0 on $T^*(\mathbb{C}^2 - \{0\})$ has formally the same expression which we rewrite as $\rho dz + \bar{\rho} d\bar{z}$ (matrix multiplication of a row by a column vector being understood). The group $U(1)$ defines a Hamiltonian action on $T^*(\mathbb{C}^2 - \{0\})$ with momentum map $J(z, \rho) = i(\rho z - \overline{\rho z}) = - 2 \, Im \, (\rho z)$.

We show next that the Legendre map of the Lagrangian $L(z, v) = \frac{1}{2} g_z(v, v)$ maps $T(\mathbb{C}^2 - \{0\})$ onto the zero level set of this momentum map.

The Legendre map corresponding to L is given by

$$FL(z, v) = \left(z, \frac{1}{2} \left(|z|^2 v^\dagger + i \, Im \, \langle z, v \rangle z^\dagger \right) \right).$$

Denoting the right-hand side by (z, ρ) we have

$$\rho z = \frac{1}{2} \left(|z|^2 \langle v, z \rangle + i \, Im \, \langle z, v \rangle |z|^2 \right) =$$

$$= \frac{1}{2} |z|^2 \, Re \, \langle z, v \rangle = \overline{\rho z}$$

and thus $J(FL(z, v)) = 0$, which is in agreement with lemma 3.2. Conversely, suppose $(z, \rho) \in J^{-1}\{0\}$ and put $v = 2|z|^{-2} \rho^\dagger + ikz$ for any real k, then

$FL(z, v) = (z, \rho)$ and so we have proven that $\mathcal{I}m\ FL = J^{-1}\{0\}$.

Finally, it may be shown that the map

$$\hat{\Sigma} : J^{-1}\{0\} \to T^*(E^3 - \{0\}), (z, \rho) \to \left(\frac{1}{2} z^\dagger \underline{\sigma} z, 2|z|^{-2} \mathcal{R}e(\rho \underline{\sigma} z) \right)$$

is a surjective fibre isomorphism. The pull-back $\hat{\Sigma}^* \tilde{\theta}_0$ of the canonical 1-form $\tilde{\theta}_0$ on $T^*(E^3 - \{0\})$ is just the restriction of θ_0 to $J^{-1}\{0\}$ and quotienting by the action of $U(1)$, which of course leaves $J^{-1}\{0\}$ invariant, gives a symplectomorphism of $J^{-1}\{0\}/U(1)$ with $T^*(E^3 - \{0\}) \cong T^*(\mathbb{C}^2 - \{0\}/U(1))$.

The regularization space is obtained by restoring the origin in the spinor space.

4.3. Time-dependent Lagrangian systems

Time-dependent Lagrangian theory is sometimes studied in a homogeneous formalism.

Suppose that L is a (possibly) time-dependent Lagrangian, i.e. a function on $TQ \times \mathbb{R}$, which is nondegenerate in the sense that the Cartan 1-form $\theta_L = L\,dt + \partial L/\partial v^i\ (dq^i - v^i\,dt)$ defines a contact structure on that odd-dimensional manifold. This also means that for each fixed t the Lagrangian L_t defined on TQ by $L_t(q, v) = L(q, v, t)$, is nondegenerate.

A corresponding homogeneous Lagrangian $\bar{L}$ is defined on $T(Q \times \mathbb{R})$ by

$$\bar{L}(q, t, v, w) = wL(q, v/w, t)$$

with (q^i, t, v^i, w) the natural bundle coordinates and where, strictly speaking, the submanifold $w = 0$ must be deleted from $T(Q \times \mathbb{R})$. Then $\bar{L}$ is homogeneous of degree 1 in the fibre coordinates.

We may describe this construction more geometrically as follows. Let M denote the manifold $T(Q \times \mathbb{R})$ with the zero set of w removed. Define a smooth map $p : M \to TQ \times \mathbb{R}$, by $p(q^i, t, v^i, w) = (q^i, v^i/w, t)$. Then

$$\bar{L} = w(p^*L).$$

Let $\bar{S}$ denote the almost tangent structure tensor field on $T(Q \times \mathbb{R})$. Then for any 1-form α on $TQ \times \mathbb{R}$

$$(p^*\alpha) \circ \bar{S} = \frac{1}{w} p^*(\alpha \circ S)$$

where S now denotes the type (1, 1) tensor field on $TQ \times \mathbb{R}$ given by

$$S = \frac{\partial}{\partial v^i} \otimes (\mathrm{d}q^i - v^i\,\mathrm{d}t).$$

It follows that

$$\begin{aligned}\theta_{\overline{L}} = \mathrm{d}\overline{L} \circ \overline{S} &= ((p^*L)\,\mathrm{d}w + w(p^*\mathrm{d}L)) \circ \overline{S} = \\ &= (p^*L)\,\mathrm{d}t + p^*(\mathrm{d}L \circ S) = \\ &= p^*\theta_L.\end{aligned}$$

Consequently,

$$\omega_{\overline{L}} = p^*\omega_L\ (= p^*\mathrm{d}\theta_L).$$

The dynamical vector field on $TQ \times \mathbb{R}$, associated with the given Lagrangian L, is the unique vector field Γ satisfying

$$\Gamma \,\lrcorner\, \omega_L = 0, \quad \Gamma(t) = 1.$$

At each point x in M, p_* is a surjective map of tangent spaces, whose kernel is a 1-dimensional subspace of the vertical subspace of T_xM and is in fact the subspace spanned by $\overline{\Delta}(x)$, where $\overline{\Delta}$ is the dilation field on $T(Q \times \mathbb{R})$.

Thus on M, char $\omega_{\overline{L}}$ is 2-dimensional: it projects under p onto the 1-dimensional distribution spanned by Γ and its vertical part is spanned by $\overline{\Delta}$.

So, dim [char $\omega_{\overline{L}}$] = 2 dim [V(char $\omega_{\overline{L}}$)] and thus $\overline{L}$ is clearly of type II. However, char $\omega_{\overline{L}}$ does not contain any vertical or any complete lift of any vector field on $Q \times \mathbb{R}$, as can be easily verified. Consequently, the homogeneous Lagrangian $\overline{L}$ is an example of a type II Lagrangian whose characteristic distribution is not a tangent distribution. Furthermore, we have

$$\mathcal{L}_{\overline{\Delta}}\overline{S} = -\overline{S},$$

and it is therefore not the case that Im $(\mathcal{L}_Z\overline{S}) \subseteq$ char $\omega_{\overline{L}}$ for each $Z \in$ char $\omega_{\overline{L}}$ since the image of $-\overline{S}$ is the whole vertical distribution of $T(Q \times \mathbb{R})$. This is in agreement with the results of section 2.5 where it was pointed out that the only type II Lagrangians L for which the almost tangent structure is projectable under char ω_L are those for which char ω_L is a tangent distribution.

On the other hand it can be shown, however, that the homogeneous Lagrangian $\overline{L}$ admits a global dynamics. Indeed since the homogeneity implies that $\overline{\Delta}(\overline{L}) = \overline{L}$, the energy $E_{\overline{L}}$ is identically zero and since char $\omega_{\overline{L}}$ is not empty, there certainly are solutions to the dynamical equation $X \,\lrcorner\, \omega_{\overline{L}} = -\mathrm{d}E_{\overline{L}}$. It follows from the fact that $\overline{L}$ is of type II that we can choose X to be a second-order equation field.

In this case the degenerate Lagrangian system is reducible in a certain sense, although not in the sense that is the main object of study in this paper. Since $\overline{\Delta}(w) = w \neq 0$ on M, we may ensure by subtracting a multiple of $\overline{\Delta}$ if necessary, that the chosen second-order equation field in char $\omega_{\overline{L}}$, say $\overline{\Gamma}$, satisfies $\overline{\Gamma}(w) = 0$. Such $\overline{\Gamma}$ is moreover unique. Now, since char $\omega_{\overline{L}}$ is integrable,

$$[\overline{\Delta}, \overline{\Gamma}] = k\overline{\Delta} + l\overline{\Gamma},$$

for certain functions k, l on M. Evaluating both sides of this equation on the coordinate functions t, w in turn, we obtain $l = 1$, $k = 0$; thus:

$$\mathcal{L}_{\overline{\Delta}}\overline{\Gamma} = \overline{\Gamma}.$$

As one would expect, this dynamical vector field is homogeneous of degree 1 in the fibre coordinates. Moreover, we may identify $TQ \times \mathbb{R}$ with the hypersurface $w = 1$ in M; $\overline{\Gamma}$ is tangent to this hypersurface and coincides there with the dynamical vector field Γ of the original time-dependent Lagrangian L. The hypersurface $w = 1$ is transverse to the action of the one-parameter group of dilations generated by $\overline{\Delta}$. We may view factoring out by char $\omega_{\overline{L}}$ as a two stage process: first we restrict to the hypersurface $w = 1$, regaining the original time-dependent picture; and second we construct the orbit space of Γ in $TQ \times \mathbb{R}$. This will be (with luck) a symplectic manifold of dimension 2 dim Q, but there is no reason to suppose that it will have an almost tangent structure, in general, consistent with its symplectic structure.

The analysis of the canonical description of the homogeneous Lagrangian $\overline{L}$ starts with the analysis of Im $(F\overline{L})$. Clearly, Im $(F\overline{L})$ is the submanifold of $T^*(Q \times \mathbb{R})$ defined by $u + \pi^*H = 0$, where $u : T^*(Q \times \mathbb{R}) \to \mathbb{R}$ and $\pi : T^*(Q \times \mathbb{R}) \to T^*Q \times \mathbb{R}$ are the projections of $T^*(Q \times \mathbb{R}) \cong T^*Q \times \mathbb{R} \times \mathbb{R}$ onto the last factor and the first two factors, respectively. H is the function on $T^*Q \times \mathbb{R}$ which for each fixed t is the Hamiltonian corresponding to the regular Lagrangian $L_t(q, v) = L(q, v, t)$.

The study of time-dependent Hamiltonian systems using the extended phase space approach, gives a very nice interpetation of the situation (cf. [30]). To H we can associate the extended Hamiltonian $\mathcal{H} = u + \pi^*H$ on $(T^*(Q \times \mathbb{R}), \overline{\Omega}_{Q\times\mathbb{R}})$, with $\overline{\Omega}_{Q\times\mathbb{R}}$ the canonical symplectic form. $\mathcal{H}$ defines a global Hamiltonian vector field $X_{\mathcal{H}}$ which projects onto the vector field Y_H on $T^*Q \times \mathbb{R}$, determined by

$$Y_H \lrcorner \Omega_H = 0, \qquad Y_H(t) = 1,$$

where $\Omega_H = \rho^*\Omega_Q - \mathrm{d}H \wedge \mathrm{d}t$, ρ denoting the projection of $T^*Q \times \mathbb{R}$ onto T^*Q.

The level sets of the extended Hamiltonian $\mathcal{H}$ define a regular coisotropic foliation of $T^*(Q \times \mathbb{R})$, transverse to the projection π. For a given leaf $S_r =$

$= \mathcal{H}^{-1}\{r\}$, let Ω_r denote the pull-back of $\Omega_{Q \times \mathbb{R}}$ to S_r.

Then (S_r, Ω_r) is presymplectomorphic to $(T^*Q \times \mathbb{R}, \Omega_H)$ and the characteristic distribution of S_r is generated by $X_{\mathcal{H}}|_{S_r}$ (for more details, see [30]). The dynamical vector field $\overline{\Gamma}$, corresponding to the homogeneous Lagrangian $\overline{L}$, is mapped under $F\overline{L}$ onto $X_{\mathcal{H}}|_{S_0}$.

The reduced phase space of the canonical picture is given by the orbit space of $X_{\mathcal{H}}|_{S_0}$ in S_0 (or, equivalently, the orbit space of Y_H in $T^*Q \times \mathbb{R}$) and is clearly symplectomorphic to the reduced space obtained in the Lagrangian picture, as described above. We finally notice that the canonical picture corresponds to the usual reduction procedure for $(T^*(Q \times \mathbb{R}), \overline{\Omega}_{Q \times \mathbb{R}})$ provided we use as a Lie algebra the 1-dimensional algebra generated by $X_{\mathcal{H}}$, which is not the Hamiltonian lift of any vector field on $Q \times \mathbb{R}$.

4.4. Lagrangians of type III

Whereas the main body of the paper has been devoted to the study of Lagrangians of type II, we also wish to include here, for completeness, the description of a general class of Lagrangians of type III, i.e. Lagrangians for which dim (char $\omega_L) < 2$ dim $[V(\text{char } \omega_L)]$.

Let α be a 1-form on Q and let L be the function induced by α on TQ, i.e.

$$L(q, v) = \hat{\alpha}(q, v) = \langle v, \alpha_q \rangle.$$

Then $\theta_L = \tau_Q^* \alpha$ and $\omega_L = \tau_Q^* \mathrm{d}\alpha$. In order for a vector $\xi \in T_{(q,v)}(TQ)$ to belong to char ω_L it must be the case that

$$(\tau_{Q*} \xi) \lrcorner (\mathrm{d}\alpha)_q = 0,$$

from which we deduce that char ω_L contains at least all vertical vectors. If we now furthermore assume that $\mathrm{d}\alpha$ is a symplectic form on Q then char ω_L consists of the vertical tangent vectors only, i.e. char $\omega_L = V(\text{char } \omega_L) = V(TQ)$. L is then obviously of type III and so is any Lagrangian obtained by adding to it a potential function $V(q)$.

In fact, a Lagrangian of the form $L = \hat{\alpha}$ can only be of type II if $\mathrm{d}\alpha = 0$ in which case L is gauge equivalent to zero. In all other cases such a Lagrangian will be of type III.

An interesting example, from a physical point of view, is provided by the Dirac-like Lagrangian which has recently been discussed for instance by Jakubiec [31]. Taking $Q = \mathbb{R}^2$, with coordinates (q^1, q^2), this Lagrangian reads

$$L(q, v) = \frac{1}{2} (q^2 v^1 - q^1 v^2 - (q^1)^2 - (q^2)^2),$$

which is of the form $L = \hat{\alpha} - V$ with $\alpha = \frac{1}{2}(q^2 \mathrm{d}q^1 - q^1 \mathrm{d}q^2)$ and $V = \frac{1}{2}[(q^1)^2 + (q^2)^2]$. Here, $\mathrm{d}\alpha = \mathrm{d}q^2 \wedge \mathrm{d}q^1$ is the standard symplectic form on $\mathbb{R}^2$ and char ω_L is spanned by $\partial/\partial v^1$ and $\partial/\partial v^2$ as predicted.

5. SOME FINAL REMARKS

In this paper we have developed a consistent Lagrangian reduction scheme for degenerate Lagrangian systems. It has been shown that the reduction process works very nicely in the case of Lagrangians L which admit a global dynamics and for which char ω_L is a tangent distribution. A typical example of such a Lagrangian is given by the kinetic energy associated with a degenerate metric, provided the null distribution of the metric consists of Killing vector fields.

The link between the present approach to the regularization problem on the one hand, and the canonical description of systems with constraints (Dirac's theory) and the Marsden-Weinstein reduction of systems with symmetry on the other hand, has been clarified. In accomplishing this program we have imposed several restrictions on the allowed class of Lagrangians, some of which are in fact rather stringent.

The overall confinement to finite dimensional systems is mainly a matter of convenience and does not really affect the validity of the theory. Upon the necessary technical modifications of the proofs, most results should be easily extendable to the case where Q is a Banach or Hilbert manifold. Of a more fundamental nature, however, is the assumption of the existence of a global dynamics for the given Lagrangian. In the canonical picture this is reflected by the fact that only primary constraints are taken into consideration (cf. proposition 3.1). The main reason for imposing this condition is that the kind of reduction we have in mind presumes the existence of a consistent second-order equation solution to the given Euler-Lagrange equations. At least for type II Lagrangians, the existence of a global dynamics sufficies for that purpose. The second-order equation problem becomes much more involved if secondary, (tertiary, etc. . . .) constraints have to be taken into account (cf. [6]). The study of the Lagrangian reduction in these more general situation will be a topic for further reserach.

A final word is also in order here regarding the almost regularity assumption which facilitates the transition from the Lagrangian to the Hamiltonian picture. The combination of almost regularity, type II and existence of a global dynamics, manifests itself in the fact that only primary first class constraints are present (cf. section 3.1).

As pointed out already in [5], almost regularity as a global condition is seldom satisfied. Hyperregular Lagrangians are almost regular but, in general, even a regular Lagrangian need not be almost regular. In practical applications, a possible way out of the problem may then be to work locally.

APPENDIX A

Proof of proposition 2.3

We first consider the following general situation. Let $\pi : M \to N$ be a surjective submersion and let R be a type $(1,1)$ tensor field on M. By definition, R projects onto N if it is π-related to a type $(1,1)$ tensor field on N, i.e., if there exists a type $(1,1)$ tensor field U on N such that for every $m \in M$ and every $x \in T_m M$

$$\pi_*(R_m(x)) = U_{\pi(m)}(\pi_*(x)).$$

For this to be the case it is clearly necessary that for every m, $m' \in M$ with $\pi(m) = \pi(m')$ and for every $x \in T_m M$ and $x' \in T_{m'} M$ such that $\pi_*(x) = \pi_*(x')$,

$$\pi_*(R_m(x)) = \pi_*(R_{m'}(x')). \tag{A1}$$

This condition is also sufficient: for, if it holds, we may define U unambiguously by setting for every $n \in N$ and $y \in T_n N$

$$U_n(y) = \pi_*(R_m(x)),$$

where m is any point of the fiber $\pi^{-1}\{n\}$ and $x \in T_m M$ any vector such that $\pi_*(x) = y$. Then, U_n is a linear map of $T_n N$, and by using coordinates adapted to π it is easy to see that U is a smooth tensor field on N.

Now, suppose that N is the leaf space of an integrable distribution $\mathcal{D}$ on M. We then show that the above necessary and sufficient condition for R to pass to the quotient is equivalent to the conditions $R(\mathcal{D}) \subset \mathcal{D}$ and $\mathrm{Im}\,(\mathcal{L}_Z R) \subset \mathcal{D}$ for all $Z \in \mathcal{D}$.

First, if $m = m'$ and x, $x' \in T_m M$ then $\pi_*(x) = \pi_*(x')$ if and only if $x - x' \in \mathcal{D}_m$. The above condition (A1) then necessarily yields $R(\mathcal{D}) \subset \mathcal{D}$.

Second, suppose that $m' = \phi_t(m)$ where ϕ_t represents the flow generated by some vector field $Z \in \mathcal{D}$. Then $\mathcal{D}_{m'} = \phi_{t*}(\mathcal{D}_m)$ and so if $x' = \phi_{t*}(x)$ we find that $\pi_*(x') = \pi_*(x)$. In order for $\pi_*(R_{m'}(x')) = \pi_*(R_m(x))$ we then must have

$$(\phi_t^{-1})_*(R_{\phi_t(m)}(\phi_{t*}(x))) - R_m(x) \in \mathcal{D}_m.$$

If this is to hold for all t (sufficiently close to 0), for all $m \in M$ and for all $x \in T_m M$, we must have

$$\mathrm{Im}\,(\mathcal{L}_Z R) \subset \mathcal{D} \quad \text{for all} \quad Z \in \mathcal{D}.$$

Conversely, suppose that $R(\mathcal{D}) \subset \mathcal{D}$ and $\mathrm{Im}\,(\mathcal{L}_Z R) \subset \mathcal{D}$ for all $Z \in \mathcal{D}$. We show that for any $m' = \phi_t(m)$, with ϕ_t again the flow generated by some $Z \in \mathcal{D}$, and for any $x \in T_m M$, $x' \in T_{m'}M$ such that $\pi_*(x) = \pi_*(x')$, we have that (A1) holds.

Note, first of all, that it is sufficient to assume $x' = \phi_{t*}(x)$ since, if this were not so, $x' - \phi_{t*}(x) \in \mathcal{D}_{m'}$ and we know that $R(\mathcal{D}) \subset \mathcal{D}$.

Let α_m be any covector at m such that (with obvious shorthand notation) $\langle \mathcal{D}_m, \alpha_m \rangle = 0$. Then, $\langle \mathcal{D}_{m'}, (\phi_t^{-1})^*(\alpha_m) \rangle = \langle \mathcal{D}_m, \alpha_m \rangle = 0$. So, if we set $\alpha_{m'} = (\phi_t^{-1})^*(\alpha_m)$ then $\langle \mathcal{D}_{m'}, \alpha_{m'} \rangle = 0$. Let X and α be the vector field and covector field obtained by Lie transporting x and α_m, respectively, along the integral curve of Z through m, and consider the function $\langle R(X), \alpha \rangle$ along that integral curve. We have

$$
\begin{aligned}
Z(\langle R(X), \alpha \rangle) &= \langle \mathcal{L}_Z(R(X)), \alpha \rangle && \text{since} \quad \mathcal{L}_Z \alpha = 0, \\
&= \langle (\mathcal{L}_Z R)(X), \alpha \rangle && \text{since} \quad \mathcal{L}_Z X = 0, \\
&= 0 && \text{since} \quad \mathrm{Im}\,(\mathcal{L}_Z R) \subset \mathcal{D}.
\end{aligned}
$$

Consequently

$$\langle R_{m'}(x'), \alpha_{m'} \rangle = \langle R_m(x), \alpha_m \rangle$$

or, equivalently,

$$\langle (\phi_t^{-1})_*(R_{m'}(x')) - R_m(x), \alpha_m \rangle = 0.$$

This holds for all covectors α_m such that $\langle \mathcal{D}_m, \alpha_m \rangle = 0$ and therefore

$$(\phi_t^{-1})_*(R_{m'}(x')) - R_m(x) \in \mathcal{D}_m.$$

But this means that $\pi_*(R_{m'}(x')) = \pi_*(R_m(x))$, as required.

So far, m' is restricted to lie on some integral curve of $\mathcal{D}$ through m. But the leaf of $\mathcal{D}$ through m consists of all points of M which may be joined to m by a piecewise smooth curve with a finite number of segments, each of which is an integral curve of $\mathcal{D}$. This completes the proof. ∎

APPENDIX B

Proof of theorem 2.2

The first step in the proof of this theorem involves showing that under the given conditions $V(TQ) \cap \hat{\mathcal{D}}$ is spanned by vertical lifts of vector fields on Q. The essential step in proving this is carried out in the individual fibres of TQ and uses their linear (or, strictly speaking, affine) structure and so may usefully be stated in terms of a distribution on $\mathbb{R}^n$.

LEMMA. *Let $\mathcal{E}$ be a distribution on $\mathbb{R}^n$ with the property that for every constant* *vector field A on $\mathbb{R}^n$, $\mathcal{L}_A \mathcal{E} \subset \mathcal{E}$. Then $\mathcal{E}$ has a basis of constant vector fields (and is thus a field of parallel subspaces of $\mathbb{R}^n$).*

Proof. Consider $\mathcal{E}$ as a vector bundle over $\mathbb{R}^n$, say E (a subbundle of $T\mathbb{R}^n$). Any vector field $Z \in \mathcal{E}$ may then be thought of as a section of E. We define a connection on E as follows: for $m \in \mathbb{R}^n$, $x \in T_m \mathbb{R}^n$ and Z a section of E defined near m,

$$\nabla_x Z = (\mathcal{L}_{\bar{x}} Z)_m,$$

where $\bar{x}$ is the constant vector field corresponding to x. Then $\nabla_x Z \in \mathcal{E}_m$ by assumption, and all the defining properties of a covariant derivative operator are clearly satisfied, except perhaps that

$$\nabla_{fY} Z = f \nabla_Y Z,$$

for any vector field Y and function f. But

$$\begin{aligned}(\nabla_{fY} Z)_m &= (\mathcal{L}_{\overline{(fY)_m}} Z)_m = (\mathcal{L}_{f(m)\bar{Y}_m} Z)_m = \\ &= f(m)(\nabla_Y Z)_m,\end{aligned}$$

as required.

Since the curvature is tensorial it is sufficient to compute $R(X, Y)Z$ when $X = A$ and $Y = B$ are constant vector fields. For any constant vector field A we have $\nabla_A Z = \mathcal{L}_A Z$ and therefore

$$\begin{aligned}R(A, B)Z &= \nabla_A \nabla_B Z - \nabla_B \nabla_A Z = \\ &= \mathcal{L}_A \mathcal{L}_B Z - \mathcal{L}_B \mathcal{L}_A Z = \\ &= \mathcal{L}_{[A,B]} Z = \\ &= 0.\end{aligned}$$

The connection is therefore flat and so there exists a parallel field of frames, i.e. vector fields $Z_1, Z_2, \ldots, Z_k \in \mathcal{E}$ ($k = \dim \mathcal{E}$) such that $\nabla_X Z_i = 0$ for all X. Thus in particular $\mathcal{L}_A Z_i = 0$ for every constant vector field A and so the vector fields Z_i are themselves constant. Then $\mathcal{E}$ is the field of parallel subspaces spanned at each point of $\mathbb{R}^n$ by the Z_i. ∎

Remark. It is perhaps worth noticing that the previous lemma is a particular case of a more general result: if a Lie group G acts simply and transitively on a manifold M and if there is a distribution $\mathcal{E}$ on M such that $\mathcal{L}_{X_a} \mathcal{E} \subset \mathcal{E}$ for every infinitesimal generator X_a (where $a \in \mathfrak{g}$), then there is a basis for $\mathcal{E}$ which is invariant under the action of G. This result may be proved by adapting the proof of the lemma, observing that the constant vector fields are the infinitesimal generators of the action of $\mathbb{R}^n$ onto itself by translation.

COROLLARY. *Let $\hat{\mathcal{D}}_V$ be a distribution of vertical subspaces on TQ such that* $\mathrm{Im}\,(\mathcal{L}_Z S) \subset \hat{\mathcal{D}}_V$ *for every* $Z \in \hat{\mathcal{D}}_V$. *Then there is a distribution $\mathcal{D}$ on Q of the same dimension as $\hat{\mathcal{D}}_V$ such that $\hat{\mathcal{D}}_V$ is spanned by the vertical lifts of vector fields in $\mathcal{D}$.*

Proof. For any vector field X on TQ and $Z \in \hat{\mathcal{D}}_V$,

$$(\mathcal{L}_Z S)(X) = [Z, S(X)] - S([Z, X]) \in \hat{\mathcal{D}}_V.$$

In particular, if X is projectable then $[Z, X]$ is vertical (since Z is) and $S(X)$ is a vertical lift, and thus $[Z, S(X)] \in \hat{\mathcal{D}}_V$. Moreover, every vertical lift is of the form $S(X)$ for some projectable X. It follows that $\mathcal{L}_{Y^v} \hat{\mathcal{D}}_V \subset \hat{\mathcal{D}}_V$ for every vertical lift Y^v.

Applying the previous lemma in each fibre, we see that $\hat{\mathcal{D}}_V$ restricted to the fibre over $m \in Q$ is spanned by vertical lifts of some $\dim(\hat{\mathcal{D}}_V)$-dimensional subspace of $T_m Q$, which defines $\mathcal{D}_m$. ■

We can now give the proof of theorem 2.2.

From the formula

$$(\mathcal{L}_Z S)(X) = [Z, S(X)] - S([Z, X])$$

we see that if Z is vertical, so is $(\mathcal{L}_Z S)(X)$ for any vector field X. Thus the corollary applies with $V(TQ) \cap \hat{\mathcal{D}}$ for $\hat{\mathcal{D}}_V$ and shows that there is a distribution $\mathcal{D}$ on Q of dimension $\frac{1}{2} \dim \hat{\mathcal{D}}$ such that $V(TQ) \cap \hat{\mathcal{D}}$ is spanned by the vertical lifts of $\mathcal{D}$. Now, at each point $x \in TQ$, the map $S_x : \hat{\mathcal{D}}_x \to \hat{\mathcal{D}}_x$ has its image contained in $V_x(TQ) \cap \hat{\mathcal{D}}_x$ and this is also its kernel from which it follows, by the assumption on the dimensions, that S_x is a surjection onto $V_x(TQ) \cap \hat{\mathcal{D}}_x$. Consequently, for any vertical vector field Z, belonging to $V(TQ) \cap \hat{\mathcal{D}}$, there is some vector field $Y \in \hat{\mathcal{D}}$ such that $S(Y) = Z$. In particular, take $Z = X^v$ where $X \in \mathcal{D}$. Then, there is a vector field $Y \in \hat{\mathcal{D}}$ such that $S(Y) = X^v$ and so, Y must be projectable and projects onto X.

Consider now the second-order equation field Γ for which $\mathcal{L}_\Gamma \hat{\mathcal{D}} \subset \hat{\mathcal{D}}$. For any $Y \in \hat{\mathcal{D}}$ we have

$$(\mathcal{L}_\Gamma S)(Y) = [\Gamma, S(Y)] - S([\Gamma, Y]) \in \hat{\mathcal{D}},$$

since $[\Gamma, S(Y)]$ and $S([\Gamma, Y])$ both belong to $\hat{\mathcal{D}}$.

Thus the projection operators $\hat{Q} = \frac{1}{2}(I + \mathcal{L}_\Gamma S)$ and $\hat{P} = \frac{1}{2}(I - \mathcal{L}_\Gamma S)$, corresponding to the vertical and horizontal decomposition defined by the Ehresmann connection generated by Γ (see e.g. [8]), map $\hat{\mathcal{D}}$ to itself. Hence, for every vector field $Y \in \hat{\mathcal{D}}$, the horizontal projection $\hat{P}(Y)$ also belongs to $\hat{\mathcal{D}}$. In particular, given $X \in \mathcal{D}$ there is a vector field on $\hat{\mathcal{D}}$ which projects onto X and its horizontal part also belongs to $\hat{\mathcal{D}}$; but this is just X^h, the horizontal lift of X, where

$$X^h = \frac{1}{2}\{([X^v, \Gamma] + X^c)\}$$

or

$$X^c = 2X^h - [X^v, \Gamma]$$

(cf. [8]). Both terms on the right-hand side belong to $\hat{\mathcal{D}}$ and therefore so does X^c. We have thus shown that $\hat{\mathcal{D}}$ contains the vertical and the complete lifts of the vector fields in $\mathcal{D}$. Since $\dim \hat{\mathcal{D}} = 2 \dim \mathcal{D}$ we see that $\hat{\mathcal{D}}$ is indeed the tangent distribution of $\mathcal{D}$ and, as pointed out before (cf. section 2.2), integrability of $\hat{\mathcal{D}}$ implies integrability of $\mathcal{D}$ (and vice versa). ■

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